Chromatic Number of in-Regular Types of Halin Graphs

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Abstract: A Halin graph \( H \) is the union of a tree \( T \neq K_2 \) with no vertex of degree two and a cycle \( C \) connecting the end-vertices of \( T \) in the cyclic order determined by a plane embedding of \( T \). In this paper, we classify the Halin graphs depending upon whether the tree \( T \) is unicentric or bicentric and investigate the vertex coloring properties of four classes of Halin graphs.

Keywords: In-regular circular Halin graph, in-regular belted circular Halin graph, in-regular elliptical Halin graph, in-regular belted elliptical Halin graph.

I. INTRODUCTION

A Halin graph is a plane graph \( G = T \cup C \) where \( T \neq K_2 \) is a tree with no vertex of degree 2 and \( C \) is a cycle connecting the leaves of \( T \) in the cyclic order determined by the plane embedding of \( T \). Halin Graphs belong to the family of planar 3-connected graphs and possess Hamiltonian properties. They are 1-Hamiltonian, (i.e., they are Hamiltonian) and remain so even after the removal of any single vertex as given in Bondy \cite{1}. In the recent years, many researchers have been studying the coloring \cite{3, 9, 10} and list coloring \cite{4} of Halin graphs. Recently some scholars begin to consider the adjacent vertex acyclic edge coloring of graphs \cite{2}, the adjacent vertex distinguishing edge coloring of planar graphs \cite{5} and Halin graphs \cite{7, 6}. We use the notation \( V(G) \) and \( E(G) \) for the vertex and the edge sets of \( G \) respectively.

In this paper we obtain the chromatic number of different types of Halin graphs depending upon whether it is unicentric or bicentric

A Halin graph \( G = T \cup C \) in which the tree \( T \) has one vertex as its center, the number of levels \( \ell \), the degree of inner vertices \( D \) and outer vertices obviously having degree three is called the in-regular circular Halin graph and denoted by \( H_1(\ell, D) \)

Example:\( H_1(2,5) \)

The vertex set of \( H_1(\ell, D) \) can be divided into two disjoint sets called inner nodes and outer nodes. The outer nodes are precisely the leaf nodes and inner nodes are the non-leaf nodes of \( T \). It is noted that only the outer nodes are in the cycle \( C \). The Halin graph in which the tree \( T \) has a star structure (i.e. only one non-leaf node) is called a wheel.
Let $H_1(\ell', D)$ be a Halin graph and $w$ be an inner node which is adjacent to only one other inner node. Define $C(w)$ as the set of all outer nodes adjacent to the inner node $w$. The sub graph of $H$ induced by $w \cup C(w)$ is referred as fan and $w$ is called center of this fan.

For an in-regular circular Halin graph $H_1(\ell', D)$, some of the interesting aspects are:

1) The total number of vertices in $H_1(\ell', D)$ is

$$1 + D + D(D-1)^{\ell} + D(D-1)^{\ell+1} + \ldots + D(D-1)^{\ell-1}$$

2) Every $H_1(\ell', D)$ is Hamiltonian.

3) The total number of leaves in $H_1(\ell', D)$ is $D(D-1)^{\ell-1}$.

4) The total number of fans in $H_1(\ell', D)$ is $D(D-1)^{\ell-2}$.

A proper vertex coloring of a graph $G$ is an assignment of colors to the vertices of $G$ one color to each vertex so that adjacent vertices are colored differently. The minimum number of colors required for the proper vertex coloring of the graph $G$ is called chromatic number, denoted as $\chi(G)$.

II. MAIN RESULTS

A. Vertex colorings in in-regular circular Halin Graph:

**Theorem 2.1**

In $H_1(\ell', D)$, $D > 2$ with level $\ell' = 1$, the chromatic number $\chi(H_1(\ell', D)) = \begin{cases} 4 & \text{if } D \text{ is odd} \\ 3 & \text{if } D \text{ is even} \end{cases}$

**Proof.**

In this case, $H_1(\ell', D) = a$ wheel. The result follows.

**Theorem 2.2**

In $H_1(\ell', D)$ where $D > 2$ with level $\ell' \geq 2$, the chromatic number $\chi(H_1(\ell', D)) = 3$.

**Proof.**

$H_1(\ell', D)$ is an in-regular circular Halin graph with level $\ell' \geq 2$ and degree $D > 2$ having one center $u$, say. Let the level $\ell$ of $H_1(\ell', D)$ be $n$ and degree be $D = m$. Since it is unicentric, fix the color $c_1$ on the central vertex $u$ where $\ell = 0$. The $m$ vertices at level $\ell = 1$ are independent and are adjacent to $u$. Hence these $m$ vertices receive a color $c_2$. If $\ell = k < n$, then the $m(m-1)^{k-1}$ vertices on the level $\ell = k$ are independent which can all be coloured by $c_1$ or $c_2$, whichever color is held by the vertices in the level $k-2$. Now, for any vertex $w$ at the $(n-1)^{th}$ level, the fan induced by the vertices $w \cup C(w)$ requires three colors, that is a color $c_3$ in addition to the two colors $c_1$ and $c_2$ already used. In this process, the vertices in the cycle $C$ are alternatively colored with two colors other than that color assigned to $w$. Since the number of vertices $m(m-1)^{n-1}$ is even for any $m$, these two colors will exhaust all the leaf nodes on the cycle. This gives $\chi(H_1(\ell', D)) \leq 3$. Since $H_1(\ell', D)$ has a triangle as an induced sub graph, $\chi(H_1(\ell', D)) \geq 3$. Hence the result follows.

A Halin graph $H_1(\ell', D)$ in which the vertices of each level $0 < \ell < n$ are connected by a cycle, contributing degree 2 to each inner vertex such that the resulting graph maintains the inner degree $D$ is called in-regular belted circular Halin graph and denoted by $BH_1(\ell', D)$.

Example: $BH_1(2, 5)$
The in-regular belted circular Halin graph $BH_1(\ell, D)$ obviously holds the following interesting properties.

1) The total number of vertices in $BH_1(\ell, D)$, $D > 3$ is
   \[1 + D + D(D - 3) + D(D - 3)^2 + \ldots + D(D - 3)^{\ell - 1}\] for $\ell \geq 1$
   1 for $\ell = 0$

2) The total number of leaves in $BH_1(\ell, D)$, $D > 3$ is $D(D - 3)^{\ell - 1}$.

3) The total number of fans in $BH_1(\ell, D)$, $D > 3$ is $D(D - 3)^{\ell - 2}$.

4) $BH_1(1, D)$ is same as $H_1(1, D)$, where $D > 2$.

**Theorem 2.3**

$BH_1(\ell, 3)$, where $\ell \geq 2$ does not exist.

**Proof.**

It is obvious that $BH_1(1, 3) = H_1(1, 3) = Wheel W_3$. If $\ell \geq 2$, then in the in-regular belted circular Halin graph with uni-center, every vertex in any level $k$, where $0 < k < \ell$, has degree 4 which is a contradiction since $D = 3$.

**Remark:**

The graph $BH_1(\ell, D)$, where $D$ is of odd and $D > 2$ with level $\ell = 1$ is a wheel on odd cycle for which the chromatic number $\chi(BH_1(\ell, D)) = 4$. In the case when $D$ is even, $\chi(BH_1(\ell, D)) = 3$.

**Theorem 2.4**

For the graph $BH_1(\ell, D)$ where $D$ is of even degree, $D > 2$ with level $\ell > 1$, the chromatic number $\chi(BH_1(\ell, D)) = 3$.

**Proof.**

Obviously $\chi(BH_1(\ell, D)) \geq 3$ since it has a triangle as an induced sub graph. Fix a color $c_1$ for the central vertex $u$. The vertices at the first level can be alternatively colored with $c_2$ and $c_3$. Each inner vertex $v$ emanates exactly $D - 3$ vertices which are belted (that is, which lie in a cycle on $D(D - 3)^k$ vertices for some $k$), where $D - 3$ is odd. The vertex $v$, together with the $(D - 3)$ vertices adjacent to $v$, induces a fan. To achieve a 3-coloring, the $(D - 3)$ vertices in any such fan in the inner cycle must have colors in one of the following schemes:

a) $c_1, c_2, c_1, c_2, \ldots, c_1, c_2, c_1$.

b) $c_2, c_3, c_2, c_3, \ldots, c_2, c_3, c_2$.

c) $c_3, c_1, c_3, c_1, \ldots, c_3, c_1, c_3$.

Let $w_1, w_2, \ldots, w_{D-3}$ be the $(D - 3)$ vertices of the fan emanating from (adjacent to) $v$.

Case a.1. $v$ is a vertex in Scheme (a).

If $v$ has the color $c_1$, then $w_1, w_2, \ldots, w_{D-3}$ are colored by Scheme (b).

If $v$ has the color $c_2$, then $w_1, w_2, \ldots, w_{D-3}$ are colored by Scheme (c).

Case b.1. $v$ is a vertex in Scheme (b).

If $v$ has the color $c_2$, then $w_1, w_2, \ldots, w_{D-3}$ are colored by Scheme (c).

If $v$ has the color $c_3$, then $w_1, w_2, \ldots, w_{D-3}$ are colored by Scheme (a).

Case c.1. $v$ is a vertex in Scheme (c).

If $v$ has the color $c_3$, then $w_1, w_2, \ldots, w_{D-3}$ are colored by Scheme (a).

If $v$ has the color $c_1$, then $w_1, w_2, \ldots, w_{D-3}$ are colored by Scheme (b).

This color scheme can be extended to all the levels of the graph and a 3-coloring can be achieved.

**Theorem 2.5**

In $BH_1(\ell, D)$ where $D$ is of odd degree, $D \geq 5$ with level $\ell > 1$, chromatic number $\chi(BH_1(\ell, D)) = 4$. 


Proof:
Obviously $\chi(BH_2(\ell', D)) \geq 4$ since the union of the central vertex and the D vertices at level $\ell' = 1$ induce a wheel on odd cycle. Let $c_1$, $c_2$, $c_3$ and $c_4$ be any four colors. Fix $c_1$ for the central vertex $u$. The vertices at the first level can be alternatively colored with $c_2$ and $c_3$ and the $D^{th}$ vertex with $c_4$, since the inner cycle at level $\ell' = 1$ is of odd length. Each inner vertex $v$ emanates exactly $D-3$ vertices which are belted (that is, which lie in a cycle on $D(D-3)^k$ for some $k$), where $D-3$ is even. The vertex $v$, together with the $(D-3)$ vertices adjacent to $v$, induces a fan. To achieve a 4-coloring, the $(D-3)$ vertices in any such fan in the inner cycle can be colored by the scheme as in Theorem 2.4, but with three colors.

Hence 4-coloring can be achieved

B. Vertex colorings in in-regular Elliptical Halin Graph:

A Halin graph in which the tree has two vertices as its centers, $\ell'$ the number of levels, $D$ the degree of inner vertices and the outer vertices having degree three is called an in-regular elliptical Halin graph and denoted by $H_2(\ell', D)$.

Example: $H_2(2,5)$

The in-regular elliptical Halin graph $H_2(\ell', D)$ has the following properties.

1) The total number of vertices in $H_2(\ell', D)$ is
   $$\begin{cases} 
   2[1+(D-1)+(D-1)^2+(D-1)^3+\ldots+(D-1)^{\ell'}] & \text{for } \ell' \geq 1 \\
   2 & \text{for } \ell'=0 
   \end{cases}$$

2) Every $H_2(\ell', D)$ is Hamiltonian.

3) The total number of leaves in $H_2(\ell', D)$ is $2(D-1)^{\ell'}$.

4) The total number of fans in $H_2(\ell', D)$ is $2(D-1)^{\ell'-1}$.

Theorem 3.1

In $H_2(\ell', D)$ where $D>2$ with level $\ell' \geq 1$, the chromatic number $\chi(H_2(\ell', D)) = 3$.

Proof.

Let $H_2(\ell', D)$ be an in-regular elliptical Halin graph with $\ell' \geq 1$, $D>2$. Fix the colors $c_1$ and $c_2$ to the two centers $u$ and $v$ respectively of the underlying tree of the graph. Let $A$ be the sub tree rooted at $u$ which has level $\ell'$. Consider the sub tree $T_1 = A - uv$. $T_1$, being a tree, is 2-colorable, say with colors $c_1$ and $c_2$ up to the level $\ell'-1$. Similarly, $T_2 = B - uv$ is also 2-colorable, say with the same colors $c_1$ and $c_2$ up to the level $\ell'-1$, where $B$ is the sub tree rooted at $v$ having level $\ell'$. Then $T = T_1 \cup T_2 \cup uv$. All the leaf nodes of $T$ lie on the cycle $C$. Since every inner vertex at level $\ell'-1$, together with the $V(C)$, induces a fan which requires an additional color, say $c_3$, in addition to $c_1$ and $c_2$. Hence $\chi(H_2(\ell', D)) \leq 3$. Since $H_2(\ell', D)$ has an odd cycle as an induced sub graph, $\chi(H_2(\ell', D)) \geq 3$. Hence the result follows.

A Halin graph $H_2(\ell', D)$ in which the vertices of each level $0 < \ell' < n$ are connected by a cycle, contributing degree 2 to each inner vertex such that the resulting graph maintains the inner degree $D$ is called in-regular belted elliptical Halin graph and denoted by $BH_2(\ell', D)$.
Example: BH₂(2,5)

For an in-regular belted elliptical Halin graph, BH₂(𝓁, D), some of the interesting aspects are:
1) For 𝓁 >2 and D =3, BH₂(𝓁, D) does not exist.
2) The total number of vertices in BH₂(𝓁, D), D >3 is
   \[2[1+(D-1)+\ldots+(D-1)(D-3)^{\ell-1}]\text{ for } \ell \geq 1\]
   \[2 \text{ for } \ell =0\]
3) Every BH₂(𝓁, D) is Hamiltonian.
4) The total number of leaves in BH₂(𝓁, D) is 2(D-1)(D-3)𝓁-1.
5) The total number of fans in BH₂(𝓁, D) is 2(D-1)(D-3)𝓁-2.

**Theorem 3.2**

BH₂(𝓁, D), where D = 3 with level 𝓁≥ 2, does not exist.

**Proof:**

Let u and v be the centers of BH₂(𝓁, D). Let x ≠ u or v be any inner vertex. Since the degree of any inner vertex other than the center is at least 4 in any belted graph BH₂(𝓁, D), it concludes that BH₂(𝓁, D) does not exist.

**Theorem 3.3**

In BH₂(𝓁, D), where D >3 with level 𝓁= 1, the chromatic number χ(BH₂(𝓁, D)) = 3.

**Proof:**

Let u and v be the centers of BH₂(𝓁, D). Assign the colors c₁ and c₂ to vertices u and v respectively. Since 𝓁= 1, the center u is adjacent to D-1 outer vertices on the outer cycle which can be alternately colored with c₂ and c₁ in order. Similarly, the other center v is also adjacent to D-1 outer vertices on the outer cycle which are colored alternately with c₁ and c₃. Since there are only 2D-2 vertices on the outer cycle, the 3-coloring on the cycle is proper, which proves the result.

**Theorem 3.4**

In BH₂(𝓁, D) where D >3 with level 𝓁>1, the chromatic number χ(BH₂(𝓁, D))=3.

**Proof:**

Extending the 3-coloring of the vertices at level 𝓁=1 obtained in Theorem 3.3, to the subsequent levels, a 3-coloring is achieved for the vertices of the graph BH₂(𝓁, D). Hence χ(BH₂(𝓁, D)) = 3.

**REFERENCES**


