Improving of Bernstein type Inequality for Complex Polynomials of Degree 5 Belongs to a_2

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Abstract: **In this paper the upper bound for the derivative of 5-th degree complex polynomials norm according to a_2 and p complex polynomials norms are studied, for this kind of polynomials, the best possibilities have found. For this Brinstein Type inequalities Clement Frappier has been published a relation for complex polynomials of degree $n \geq 6$ (Theorem 8, [9]), but for n=2, 3, 4 and 5 do not exist a unique relation. I have obtained the best possibility for d_5 in the following relation.** Let $p(z) := \sum_{j=1}^{5} a_j z^j \, \epsilon P_n$; then $\|p'\| + d_5|a_2| \leq 5\|p\|$,

d_5 is the smallest positive rot of the following equation. $80 - 286x^2 + 16x^3 + 106x^4 + 12x^5 - x^6 = 0$

Keywords: **Norm Preserving, Positive Defined, Hadamard Product, Hermitian Matrix, Linear Complex Spaces, Analytic Function.**

I. INTRODUCTION

Let $p(z) := \sum_{j=1}^{n} a_j z^j \, \epsilon P_n$; with complex coefficients, where P_n is Class of all polynomials with degree at most n. Bernstein's classical inequality states $\|p'\| \leq n\|p\|$ where $\|p\| := \max_{|z|=1}|p(z)|$, from this inequality we can obtain $\|p^{(k)}\| \leq \frac{n!}{(n-k)!}\|p\|$, for $1 \leq k \leq n$. This type of inequality was improved by many mathematicians by deferent shape, in some of them seem a_0, a_2 and a_3 and some of them have trigonometric form, In this paper I have found an equality regarding to a_2 for n=5, which for $n \geq 6$ was studied by Clement Frappier.

The proof methods are "bound preserving convolution operators in the unit disk and interpolation formulas". First some definitions and useful theorems mentioned, then by the using these theorems the problem solved.

Let A be set of all analytic functions in$|z| < 1$, A_0 All function $f \in A$ with $f(0) = 1$, R Set of all functions with $\mathrm{Re}(f(z)) > \frac{1}{2}$, $\overline{co}$ Convex domain.

Definition 1:

Let $f(z) = \sum_{i=1}^{n} a_k z^k$ and $g(z) = \sum_{i=1}^{n} b_k z^k$ two analytic functions, the function the convolution (or Hadamard product) $f * g$, defined by

$$(f * g)(z) := \sum_{i=1}^{n} a_k b_k z^k$$

also belong A.

Definition 2:

i) A function $f \in A$ is norm preserving for P_n if $\|f * p\| \leq \|p\|$ for all $p \in P_n$, $\|p\| := \sup_{|z|<1}|p(z)|$. Set of these functions show with B_n (Dimiter & Richard, 2002).

ii) A function $f \in A$ is convexity preserving on P_n if $(f * p)(D) \subset co(p(D))$ for all $p \in P_n$. Set of these functions show with B_n^0.

Definition 3. A Hermitian matrix A is positive definite if $x * Ax \geq 0$ for every $x \in \mathbb{C}^n$ (Blyth & Robertson, 2006).

II. LITERATURE REVIEW

Let $p(z) := \sum_{j=1}^{n} a_j z^j \in P_n$ with complex coefficients, the famous Bernstein's inequality states that $\|p'\| \leq n\|p\|$, this kind of inequality have improved by Ruschewey $\|p'\| + \frac{2n}{n+2}|a_0| \leq n\|p\|$. Then Fournier has published a deferent relation which belong to n and $'$, $\left|p(z) - \frac{zp\prime(z)}{n}\right| + \left|\frac{zp\prime(z)}{n}\right| \leq |p|_D, z \in \bar{D}, p \in P_n$, where D is unit disk (Fournier, 2004). Khavagpur found an inequality regarding to k-th derivative (Khavagur, 2000).

$$|p^{(k)}(\beta)| \leq \frac{n(n-1)(n-2)\dots(n-k+1)}{|\beta|^k}\left[\frac{1}{2^k}\left\{|p(\beta)| + \max_{1\leq i\leq n}|p(\beta z_i)| + \left(1-\frac{1}{2^k}\right)\max_{1\leq t\leq 2n}|p(\beta z_t)|\right]\right., \beta \neq 0, k \geq 1,$$

Theorem: A Hermitian Matrix

$$A := \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix}, \quad a_{ij} = \bar{a}_{ji}$$

is positive definite if it's all principle minors $k \leq n$,

$$A_k := \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1k} \\ a_{21} & a_{22} & \cdots & a_{2k} \\ \vdots & \vdots & \ddots & \vdots \\ a_{k1} & a_{k2} & \cdots & a_{kk} \end{pmatrix}$$

are positive definite (Fuzhen, 2009).

Definition: Let V and W linear complex spaces, a function $f: V \times W \to \mathbb{C}$ on $V \times W$ with the following properties is semi bilinear, for all $k_1, k_2 \in \mathbb{C}$ and all $x_1, x_2, x \in V$ and $\eta_1, \eta_2 \in W$ the

$$f(k_1x_1 + k_2x_2, \eta) = k_1 f(x_1, \eta) + k_2 f(x_2, \eta)$$

$$f(x, k_1\eta_1 + k_2\eta_2) = \bar{k}_1 f(x, \eta_1) + \bar{k}_2 f(x, \eta_2).$$

if $V = W$ then f is semi bilinear on V (Steven, 2008).

Definition: Let V be a complex vector space, a function $g: V \to \mathbb{C}$ on V is Hermitian, if a Hermitian semi bilinear form $f: V \to \mathbb{C}$ exist such that $g(x) = f(x, x)$ for all $x \in V$.

Let V be n dimensional vector space and $\tilde{B} := \{a_1, a_2, \dots, a_n\}$ a basis for V and $g: V \to \mathbb{C}$ Hermitian form on V. Let $A := (a_{ij}) = (f(a_j, a_i)) \in \mathbb{C}^{n\times n}$ be matrix form f then for every $x \in V$ component vector $x := (x_1, x_2, \dots, x_n)^T \in \mathbb{C}^n$ we have,

$$g(x) = f(x, x) = f(\sum_{k=1}^{n} x_k a_k, \sum_{\mu=1}^{n} x_\mu a_\mu) = \sum_{k=1}^{n}\sum_{\mu}^{n} x_k \bar{x}_\mu f(a_k, a_\mu) = x * Ax$$

A is matrix form g, belong to basis $\tilde{B}$ (Seymour & Marc, 2004).

III. METHODOLOGY

The proof methods are "bound preserving convolution operators in the unit disk and interpolation formulas". By the help of theorem.1 and conservative prosperities' of B_n^0 ($Q(z) \in B_n^0$ then $\|Q * p\| \le \|p\|$ ", and $Q \in B_n^0 \Leftrightarrow Q^* \in B_n^0$,) have found a related $Q \in B_n^0$ polynomial. For this propose have the related coefficients determinant of $Q(z)$, analyzed in which area this determinant and its principle minors are positive. For the destining of this polynomial $Q \in B_n^0$ the theorem 1 and theorem 7 are used.

A. Some useful Theorems

Theorem 1: Let V be a n dimensional complex vector space and g be a Hermitian form on V, then g is positive defined if and only if the diagonal element of matrix form of g are positive (Gerd, 2011).

Proof. Lets g define by,

$$g(x) := x^* Ax = \sum_{\mu=1}^{n} \sum_{\nu=1}^{n} a_{\mu\nu} \bar{x}_\mu x_\nu .$$

for every $x \in \mathbb{C}^n$ and f is related polar form. Then A is matrix of g belong to standard basis $\{e_1, e_2, \dots, e_n\}$, $e_i \in \mathbb{C}^n$, $i \in I_n := \{1, 2, \dots, n\}$.

First Lets for every $k = 1, 2, \dots, n$, the $A_k \neq 0$. (1)

We will show that exist a basis $\{b_1, b_2, \dots, b_n\}$, $b_i \in \mathbb{C}^n$, $i \in I_n$ such that matrix form of g is diagonal and its diagonal elements are also belong to principle minor of this matrix.

we have vectors $b_1, b_2, \dots, b_n \in \mathbb{C}^n$ with the properties,

$f(b_\mu, b_\nu) = 0$, *for all* $\mu, \nu = 1, 2, \dots, n, \quad \mu \neq \nu$ ………………….. (2)

To make it, the following relations will help,

$$\begin{aligned} b_1 &= \beta_{11} e_1, \\ b_2 &= \beta_{21} e_1 + \beta_{22} e_2, \\ &\vdots \quad \vdots \quad \vdots \\ b_n &= \beta_{n1} e_1 + \beta_{n2} e_2 + \dots + \beta_{nn} e_n \end{aligned}$$ ……………………………….. (3)

We need that,

$f(b_\nu, e_\mu) = 0$ for all $\mu = 1, 2, \dots, \nu - 1$ and $f(b_\nu, e_\nu) = 1$ for all $\nu = 1, 2, \dots, n$ ………………….. (4)

then if $\mu < \nu$

$$f(b_\nu, b_\mu) = f(b_\nu, \sum_{\lambda=1}^{n} \beta_{\mu\lambda} e_\lambda) = \sum_{\lambda=1}^{n} \bar{\beta}_{\mu\lambda} f(b_\nu, e_\lambda) = 0.$$

For $\mu > \nu$ the same result will obtain, then f is Hermitian, also $f(b_\mu, b_\nu) = \overline{f(b_\mu, b_\nu)} = 0$ for all $\mu \neq \nu$.

From (4) obtains (2). Now we have to define Vectors $b_1, b_2, \dots, b_n \in \mathbb{C}^n$ that satisfy (10).

Choose b_ν from (4) for a fix $\nu \in I_n$ (here $I_n = \{1, 2, \dots, n\}$) in (3), it give a non-homogenous linear equation system,

$$f(\sum_{\lambda=1}^{\nu} \beta_{\nu\lambda} e_{\lambda\mu}, e_\nu) = 0, \quad for\ \mu = 1, 2, \dots, \nu - 1$$

$$f(\sum_{\lambda=1}^{\nu} \beta_{\nu\lambda} e_{\lambda\mu}, e_\nu) = 1$$

$\beta_{v1}, \beta_{v2}, ..., \beta_{vv}$ are unknowns with the properties that semi bilinear form f,

$$\sum_{\lambda=1}^{v} \beta_{v\lambda} f(e_\lambda, e_\mu) = 0, \quad for\ \mu = 1, 2, ..., v-1$$

$$\sum_{\lambda=1}^{v} \beta_{v\lambda} f(e_\lambda, e_v) = 1$$

and for $\mu, v \in I_n$, $f(e_v, e_\mu) = a_{\mu v}$

$$\begin{aligned} &a_{11}\beta_{v1} + a_{12}\beta_{v2} + ... + a_{1v}\beta_{vv} = 0 \\ &a_{21}\beta_{v1} + a_{22}\beta_{v2} + ... + a_{2v}\beta_{vv} = 0 \\ &\vdots \qquad \vdots \qquad \ddots \\ &a_{v-11}\beta_{v1} + a_{v-22}\beta_{v2} + ... + a_{v-1v}\beta_{vv} = 0 \\ &a_{v1}\beta_{v1} + a_{v2}\beta_{v2} + ... + a_{vv}\beta_{vv} = 1 \end{aligned} \qquad \text{.......................... (5)}$$

Determinant of the coefficient of this non-homogenous linear system is A_v, it is under (1) not zero. Then (5) have solution $\beta_{v1}, \beta_{v2}, ..., \beta_{vv}$ for every $v \in I_n$ and $b_1, b_2, ..., b_n$ vectors which by (3) uniquely obtained. More over these vectors are linear independent. From $b := \sum_{v=1}^{n} \lambda_v b_v = 0$ obtain,

$$f(b, e_1) = \lambda_1 f(b_1, e_1) + \lambda_2 f(b_2, e_1) + ... + \lambda_n f(b_n, e_1) = 0,$$

From (3), $\lambda_1 = 0$, and from $f(b, e_2) = 0$, will give $\lambda_2 = 0$, …, from $f(b, e_n) = 0$, $\lambda_n = 0$. Then $\{b_1, b_2, ..., b_n\}$ is a basis for $\mathbb{C}^n$.

Now if $B := (b_{\mu v})$ be matrix Form of g related to $\{b_1, b_2, ..., b_n\}$ basis. For $\mu, v \in I_n$

$$\begin{aligned} g(x) = f(x, x) &= f(\sum_{v=1}^{n} x_v a_v, \sum_{\mu=1}^{n} x_\mu a_\mu) \\ &= \sum_{v=1}^{n} x_v \sum_{\mu=1}^{n} \bar{x}_v f(a_v, a_\mu) = x^* A x, \end{aligned}$$

$b_{\mu v} = f(b_v, b_\mu)$.

From (2) $b_{\mu v} = 0$ for all $\mu, v \in I_n$ then B is diagonal. For every diagonal element β_{vv}, $v = 1, 2, ..., n$, we have

$$\begin{aligned} b_{vv} &= f(b_v, b_v) = f(\sum_{\lambda=1}^{v} \beta_{v\lambda} e_\lambda, b_v) = \sum_{\lambda=1}^{v} \beta_{v\lambda} f(e_\lambda, b_v) \\ &= \sum_{\lambda=1}^{v} \beta_{v\lambda} \overline{f(b_v, e_\lambda)} = \beta_{vv} \end{aligned}$$

If $A_0 := 1$, from (5) and Cramer role,

$\beta_{vv} = \dfrac{A_{v-1}}{A_v}$, for $v = 1, 2, ..., n$.

Hence,

$$B = dia(\frac{A_0}{A_1}, \frac{A_1}{A_2}, ..., \frac{A_{n-1}}{A_n}) \qquad \text{.................................... (6)}$$

is the form matrix g related to basis $\{b_1, b_2, ..., b_n\}$.

Let the principle minors A_v of A are positive definite, then $\frac{A_{v-1}}{A_v} > 0$ for all $v = 1, 2, ..., n$ (due to $A_0 = 1$ also $v = 1$).

From (6) g also A are positive definite.

Now Let A and g are positive definite. We will show that $A_v \neq 0$ for all $v \in I_n$,

$$A_v := \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1v} \\ a_{21} & a_{22} & \cdots & a_{2v} \\ \vdots & \vdots & \ddots & \vdots \\ a_{v1} & a_{v2} & \cdots & a_{vv} \end{vmatrix} = \begin{vmatrix} f(e_1, e_1) & f(e_2, e_1) & \cdots & f(e_v, e_1) \\ f(e_1, e_2) & f(e_2, e_2) & \cdots & f(e_v, e_2) \\ \vdots & \vdots & \ddots & \vdots \\ f(e_1, e_v) & f(e_2, e_v) & \cdots & f(e_v, e_v) \end{vmatrix} = 0$$

The columns of this determinant are linear defendant, therefore exist scholars $\lambda_1, \lambda_2, \ldots, \lambda_v \in \mathbb{C}$ with $(\lambda_1, \lambda_2, ..., \lambda_v) \neq (0, 0, ..., 0)$ and $\lambda_1 f(e_1, \mu) + \lambda_2 f(e_2, \mu) + ... + \lambda_v f(e_v, \mu) = 0$, for $\mu = 1, 2, ..., v$.

Hence $f(\lambda_1 e_1 + \lambda_2 e_2 + ... + \lambda_v e_v, \mu) = 0$ implies,

$f(\lambda_1 e_1 + \lambda_2 e_2 + ... + \lambda_v e_v) = 0$ ………………………………. (7)

Now $x_0 := \lambda_1 e_1 + \lambda_2 e_2 + ... + \lambda_v e_v$ due to $(\lambda_1, \lambda_2, ..., \lambda_v) \neq (0, 0, ..., 0)$ is unequal to zero, but $g(x_0) = 0$. From this contradiction A is positive definite the assumption that $A_v = 0$, $v \in I_n$ is not true then $A_v \neq 0$ for all $v \in I_n$. From the first step of proof exist a basis $\{b_1, b_2, ..., b_n\}$ for $\mathbb{C}^n$ relative that $diag(\frac{A_0}{A_1}, \frac{A_1}{A_2}, \ldots, \frac{A_{n-1}}{A_n})$ is form matrix of g. Then $\frac{A_{v-1}}{A_v} > 0$ for all $v \in I_n$ implies $A_1 > 0, \quad A_2 > 0, ..., A_n > 0$.

Theorem 2: $f \in B_n$ If and only if a complex mas μ with $\|\mu\| \leq 1$ and an analytic function f on ∂D_1 exists such that $f(z) = \int_{\partial D_1} \frac{1}{1 - \zeta z} d\mu + z^{n+1} F(z)$ (Ruscheweyeh, 1982).

Note: From the theorem.3, $f \in B = \cup_n B_n$ if and only if, a complex mass μ with $\|\mu\| \leq 1$ exist such that

$$f(z) = \int_{\partial D} \frac{1}{1 - z\zeta} d\mu(\zeta), \ z \in D_1$$

if $f \in B_n$ and $f(0) = 1$, then μ is probability mass. In this case,

$$(f * q)(z) = \int_{\partial D} q(\zeta) d\mu(\zeta) \in \overline{co}\, q(D_1)$$

and f is as well convexity obtained. From the other said, if f is convexity obtained on P_n, then have to be $f(0) = 1$ (from the $q \equiv 1 \in P_n$) and $f \in B_n$ satisfy.

Theorem 3: $f \in A$ Is convexity preserving on P_n, if a probability mass μ on ∂D_1 and $F \in A$ exist such that

$$f(z) = \int_{\partial D_1} \frac{1}{1 - z\zeta} d\mu + z^{n+1} F(z).$$

From a famous Herglotz theorem is clear that the set of

$$f(z)=\int_{\partial D_1}\frac{1}{1-z\zeta}d\mu,\ \mu$$

is probability mass, functions are equal to R (Ruscheweyeh, Convolutionin geometric function theory, 1982).

Theorem 4: The following statements are equivalent:

$f\in B_n^0$.

$\overline{co}[(f*g)D_1]\subset\overline{co}(D_1),\ q\in P_n$.

$h\in R$, $F\in A$, exist such that $f=h+z^{n+1}F$.

Lemma 1: A polynomial $Q\in P_n$ belongs to B_n^0 if and only if, exist a $f\in R$ with the following properties $f(z)-Q(z)=O(z^n)$ for $z\to 0$.

Theorem 5: Let $f(z):=\frac{a_0}{2}+\sum_{n=1}^{\infty}a_nz^n,(a_0\in \mathrm{R})$. Then $f(z)$ is holomorphic and $\mathrm{Re}(f(z))\geq 0$ for $|z|<1$ if and only if $A_n>0,(n=0,1,2,\ldots)$ or $A_0>0,A_1>0,\ldots,A_{k-1}>0,A_k=A_{k+1}=\ldots=0,$ the A_n in theorem.1 defined (Tsuji, potential teory in modern function theory, 1959).

Theorem 6: $Q(z):=1+\sum_{n=1}^{\infty}a_nz^n\in B_n^0$ If and only if, when the following hermit's matrix is positive defined (Dimiter & Richard, 2002).

$$A_n(Q)=\begin{bmatrix}1 & a_1 & a_2 & \cdots & a_n\\ \overline{a_1} & 1 & a_1 & \cdots & a_{n-1}\\ \vdots & \vdots & \vdots & \ddots & \vdots\\ \overline{a_{n-1}} & \overline{a_{n-2}} & \overline{a_{n-3}} & \cdots & a_1\\ \overline{a_n} & \overline{a_{n-1}} & \overline{a_{n-2}} & \cdots & 1\end{bmatrix}$$

Proof: If $Q\in B_n^0$, then exist $f\in R$, such that $f(z)-Q(z)=O(z^n)$. By using the theorem.6 $A_n(Q)$ have to be positive semi defined. Conversely if this is not true then the $A_n(Q)$ is positive semi defined, by using theorem 6, the developing of $Q(z)$ to a function $f\in R$ such that $f(z)-Q(z)=O(z^n)$ for $z\to 0$ and lemma (1) show that $Q\in B_n^0$.

Theorem 7: Let $p(z):=\sum_{j=0}^{n}a_jz^j\in P_n,\ n\geq 6$, then

$\|p'\|+d_n|a_2|\leq n\|p\|$.

d_n is in (0, 1) interval root of the following equation.

$$4n-(12n+4)x^2-x^3+(5n+7)x^4-\frac{5}{2}x^5-\frac{n+6}{16}x^6=0.$$

The d_n is the best possible number for $n\geq 6$ CITATION Placeholder3 \t \l 1033 (Frappier, 1988).

IV. ESTIMATION OF COMPLEX POLYNOMIALS NORMS OF DEGREE 5 AND ITS DERIVATIVES BELONGS TO a_2

By using the above information, I well find a best possibility d_n such that $\|p'\|+d_n|a_2|\leq n\|p\|$ for n=5.

Let $p(z):=\sum_{j=1}^{5}a_jz^j\ \epsilon P_n$; then

$$\|p'\|+d_5|a_2|\leq\|p\|,$$

d_5 is the smallest positive rot of the following equation.

$$32 - 200x^2 + 8x^3 + 152x^4 + 12x^5 - x^6 = 0$$

Note: for $n \geq 6$ a theorem has by Cle'ment Frappier have proofed.

Proof: From Frappier [1] we have,

$$\|p'\| + d_n|a_2| = \sup_{\alpha<d_n}\|zp'(z) + \bar{\alpha}a_2z^2\|, \text{ and } \frac{1}{n}[zp'(z) + \bar{\alpha}a_2z^2] = Q(z) * p(z),$$

$Q(z) := \frac{z}{n} + \frac{\bar{\alpha}+2}{n}z^2 + \sum_{j=3}^{n}\frac{j}{n}z^j, \quad n \geq 3.$

Then,

$$Q^*(z) = \sum_{j=3}^{n}\frac{n-j}{n}z^j + \frac{\alpha+2}{n}z^{n-2} + \frac{1}{n}z^{n-1}.$$

We will study the definiteness the following matrix,

$$m_n(\alpha) := \begin{pmatrix} n & n-1 & n-2 & \cdots & 3 & \alpha+2 & 1 & 0 \\ n-1 & n & n-1 & \cdots & 4 & 3 & \alpha+2 & 1 \\ n-2 & n-1 & n & \cdots & 5 & 4 & 3 & \alpha+2 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ \bar{\alpha}+2 & 3 & 4 & \cdots & n-1 & n & n-1 & n-2 \\ 1 & \bar{\alpha}+2 & 3 & \cdots & n-2 & n-1 & n & n-1 \\ 0 & 1 & \bar{\alpha}+2 & \cdots & n-3 & n-2 & n-1 & n \end{pmatrix}$$

We will study the following polynomial that for which α it belongs to B_n^0?

$$Q^*(z) = \sum_{j=0}^{n-3}\frac{n-j}{n}z^j + \frac{\alpha+2}{n}z^{n-2} + \frac{1}{n}z^{n-1} + 0 \cdot z^n$$

This belongs only to definiteness of the $m_n(\alpha)$ matrix. For this propose we will direct calculate belong to principle minors. The following Mathematica program is useful for calculation.

For n=5 the definiteness of the following matrix has to be studied,

$$m_5(\alpha) := m(5,5,\alpha,\bar{\alpha}) = \begin{pmatrix} 5 & 4 & 3 & \alpha+2 & 1 & 0 \\ 4 & 5 & 4 & 3 & \alpha+2 & 1 \\ 3 & 4 & 5 & 4 & 3 & \alpha+2 \\ \bar{\alpha}+2 & 3 & 4 & 5 & 4 & 3 \\ 1 & \bar{\alpha}+2 & 3 & 4 & 5 & 4 \\ 0 & 1 & \bar{\alpha}+2 & 3 & 4 & 5 \end{pmatrix}$$

The first principle minor,

$$\det(m_1(\alpha)) := \det(m(5,\alpha,\bar{\alpha})) = |5| = 5$$

The second Principle minor is,

$$\det(m_2(\alpha)) := \det(m(5,2,\alpha,\bar{\alpha})) = \begin{vmatrix} 5 & 4 \\ 4 & 5 \end{vmatrix} = 9$$

The third principle minor is,

$$\det(m_3(\alpha)) := \det(m(5,3,\alpha,\bar{\alpha})) = \begin{vmatrix} 5 & 4 & 3 \\ 4 & 5 & 4 \\ 3 & 4 & 5 \end{vmatrix} = 16$$

The third principle minor is,

$$\det(m_4(\alpha)) := \det(m(5,4,\alpha,\bar{\alpha})) = \begin{vmatrix} 5 & 4 & 3 & \alpha+2 \\ 4 & 5 & 4 & 3 \\ 3 & 4 & 5 & 4 \\ \bar{\alpha}+2 & 3 & 4 & 5 \end{vmatrix}$$

$$= 28 + 2\alpha + 2\bar{\alpha} - 9\alpha\bar{\alpha}$$

$$= 28 + 4(\mathrm{Re}(\alpha)) - 9|\alpha|^2$$

m_4 is positive if

$$p_2(x) := 28 - 4x - 9x^2 > 0$$

The smallest positive root of $p_2(x)$ is $x \cong 1.55556$.

The fourth principle minor is:

$$\det(m_5(\alpha)) = \det(m(5,5,\alpha,\bar{\alpha})) = \det\begin{pmatrix} 5 & 4 & 3 & \alpha+2 & 1 \\ 4 & 5 & 4 & 3 & \alpha+2 \\ 3 & 4 & 5 & 4 & 3 \\ \bar{\alpha}+2 & 3 & 4 & 5 & 4 \\ 1 & \bar{\alpha}+2 & 3 & 4 & 5 \end{pmatrix}$$

$$= 48 - 80|\alpha|^2 - 4 = 28 + 2\big(2Re(\alpha)\big) + |\alpha|^2 = 28 + 4Re(\alpha)$$

$$48 - 80|\alpha|^2 - 4(\alpha^2\bar{\alpha} + \alpha\bar{\alpha}^2) + 5\alpha^2\bar{\alpha}^2 = 48 - 80|\alpha|^2 - 4(\bar{\alpha} + \alpha)\alpha\bar{\alpha} + 5|\alpha|^4$$

$$= 48 - 80|\alpha|^2 - 4(2Re(\alpha))|\alpha|^2 + 5|\alpha|^4$$

is positive if,

$$p_4(x) = 48 - 80x^2 + 8x^3 + 5x^4 > 0$$

The smallest positive root of $p_4(x)$ is $x \cong 0.82842$

$$\det(m_5(\alpha)) = \det(m(5,5,\alpha,\bar{\alpha})) = \det\begin{pmatrix} 5 & 4 & 3 & \alpha+2 & 1 & 0 \\ 4 & 5 & 4 & 3 & \alpha+2 & 1 \\ 3 & 4 & 5 & 4 & 3 & \alpha+2 \\ \bar{\alpha}+2 & 3 & 4 & 5 & 4 & 3 \\ 1 & \bar{\alpha}+2 & 3 & 4 & 5 & 4 \\ 0 & 1 & \bar{\alpha}+2 & 3 & 4 & 5 \end{pmatrix}$$

$$\det\big(m_5(\alpha)\big) = 80 + 4\alpha^2 - 256\alpha\bar{\alpha} - 8\alpha^2\bar{\alpha} - 6\alpha^3\bar{\alpha} + 4\bar{\alpha}^2 - 8\alpha\bar{\alpha}^2 + 121\alpha^2\bar{\alpha}^2 - 6\alpha^3\bar{\alpha}^2 - 6\alpha\bar{\alpha}^3 - 6\alpha^2\bar{\alpha}^3 - \alpha^3\bar{\alpha}^3$$

$$= 80 + 4\alpha^2 - 256\alpha\bar{\alpha} - 8\alpha^2\bar{\alpha} - 6\alpha^3\bar{\alpha} + 4\bar{\alpha}^2 - 8\alpha\bar{\alpha}^2 + 121\alpha^2\bar{\alpha}^2 - 6\alpha^3\bar{\alpha}^2 - 6\alpha\bar{\alpha}^3 - 6\alpha^2\bar{\alpha}^3 - \alpha^3\bar{\alpha}^3$$

$$= 80 + 4(\alpha^2 + \bar{\alpha}^2) - 256|\alpha|^2 - 8(\alpha^2\bar{\alpha} + \alpha\bar{\alpha}^2) - 6(\alpha^3\bar{\alpha} + 6\alpha\bar{\alpha}^3) + 121\alpha^2\bar{\alpha}^2 - 6(\alpha^3\bar{\alpha}^2 + \alpha^2\bar{\alpha}^3) - \alpha^3\bar{\alpha}^3$$

$$= 80 + 4(4(Re(\alpha))^2 - 2|\alpha|^2) - 256|\alpha|^2 - 8(\alpha + \bar{\alpha})\alpha\bar{\alpha} - 6(\alpha^2 + \bar{\alpha}^2)\alpha\bar{\alpha} + 121\alpha^2\bar{\alpha}^2 - 6(\alpha + \bar{\alpha})\alpha^2\bar{\alpha}^2 - \alpha^3\bar{\alpha}^3$$

$$= 80 + 16(Re(\alpha))^2 - 8|\alpha|^2 - 256|\alpha|^2 - 16Re(\alpha)|\alpha|^2 - 6(4(Re(\alpha))^2|\alpha|^2 - 2|\alpha|^2)|\alpha|^2 + 121\alpha\bar{\alpha}^2 - 12Re(\alpha)|\alpha|^4 - |\alpha|^6$$

$$= 80 + 16Re(\alpha))^2 - 8|\alpha|^2 - 256|\alpha|^2 - 16Re(\alpha)|\alpha|^2 - 24(Re(\alpha))^2|\alpha|^2 + 12|\alpha|^4 + 121\alpha^2\bar{\alpha}^2 - 12Re(\alpha)|\alpha|^4 - |\alpha|^6$$

$$= 80 + 16Re(\alpha))^2 - 264|\alpha|^2 - 16Re(\alpha)|\alpha|^2 - 24(Re(\alpha))^2|\alpha|^2 + 133|\alpha|^4 - 12Re(\alpha)|\alpha|^4 - |\alpha|^6$$

is positive if

$$p(x) = 80 - 16x^2 - 264x^2 + 16x^3 + 106x^4 + 12x^5 - x^6 > 0$$

$$p(x) = 80 - 286x^2 + 16x^3 + 106x^4 + 12x^5 - x^6 > 0$$

The smallest positive root of $p_4(x)$ is $x \cong 0.57848$.

V. CONCLUSION

The above calculations shows that the smallest positive root of p_2 ($x \cong 1.55556$) is greater than the smallest positive root of p_4(is $x \cong 0.82842$) is greater then the smallest positive root of $p_6(x^* = 0.57848)$. Then m_5 is positive definite if $|\alpha|$ is smaller then the smallest positive root of p_6. It is also clear that the $p_6(x)$ for $x = x^* + \varepsilon$ can be negative for a small $\varepsilon > 0$. From the study of the d_n for $n = 5$ and for $n \geq 6$ which obtained by Fprafer obtained, that d_n is monotone increasing and for $n \to \infty$ is the smallest positive zero of. Some value of d_n is in the following table:

n	$d_n \cong$
5	0.587625
10	0.622260
20	0.627127
100	0.631069
∞	0.632062

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