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Analytical Solution of Higher Order Partial Differential Equations

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Abstract: **For many decades, separation of variable is recognized as one of the most powerful techniques for solving linear partial differential equations PDEs. The present paper proposes analytical solution for higher order homogeneous partial differential equations PDEs under specified boundary conditions BCs within a rectangular domain. Firstly, separation of variables and integral factors are used to reduce the given partial differential equation PDE to an ordinary differential equation ODE. After symbolic manipulations, a power series expansion of the unknown function is utilized to create the analytical solution. The present paper is a unique case of separation of variables which rely on eliminating one variable to solve the PDE on the other variable. The proposed closed form solution presented here reduces the effort consumed for implement the alternative numerical solutions. The effectiveness of the obtained method proves the capability to provide an analytical solution overcoming the complexity of boundary conditions and mixed derivatives in the solution of higher order linear PDE.**

Keywords: **Partial Differential Equation, separation of variables, shape function, Power Series.**

I. INTRODUCTION

Through recent decades, different implementation of higher order linear PDEs was used, beside the empirical techniques, in shape design for engineering manufacturing, or in general, for the representation of solid surfaces [1], [2]. PDEs are utilized for representation and manipulation of surface/solid models in a computer-aided geometric design. They owned fundamental importance in engineering for analysis and simulation in addition to their importance in medical sector for body tissue visualization and surgical simulation [2]-[4]. Exact solution of linear PDEs of many engineering applications still in mind the mathematicians and specialized publishers. In early 1950's, M.H. Martin [5] observed a special case of two term equation derived if the separation of variables is employed to Laplace's equation. His notation established a turning point for the traditional methods of reduction, opening up new visions to what is now known as functional separation of variables. The higher order multi-term PDE involving fractional derivative in time is studied by E. Karimov and S. Pirnafasov [6]. They used separation of variables to reduce fractional order PDE to the integer order. I.V. Rakhmelevich [7] transformed multi-dimensional PDE incorporating linear differential operator to reduced one and solved it by using separation of variable. Applying quasi-separation of variables, W. N. Everitt and B. T. Johansson [8] solved the Dirichlet problem for the biharmonic PDE on a bounded region. A. S. Berdyshev and B. J. Kadirkulov [9] studied a nonlocal problem for a fourth-order parabolic equation with the Dzhrbashyan–Nersesyan fractional differential operator. They used separation of variables to prove the existence and uniqueness of the solution of the problem. Publishers' attempts on the separation of variables of higher order PDE have already been previously done using different methods such as Fourier series and Galerkin formulation.

Highly accurate analytical solutions for many applications such as anisotropic and orthotropic rectangular plates are introduced by several researchers [10]-[12]. In the present paper, a semi analytical technique is proposed relying on applying circumvent between generalized separation of variables and integral factors. The derived technique leads to exact closed form solution for higher order PDEs incorporating mixed derivatives. The present method is applied to higher order linear PDEs with constant coefficients. The shape function is created according to the investigated PDEs and boundary conditions. On the other hand, the unknown separated function is expressed as a polynomial of power series. The derived technique is applied to establish the closed form solution of vibration of composite plate under the proposed boundary conditions BCs.

II. PARTIAL DIFFERENTIALEQUATION PDE

The general higher order homogenous PDE is:

$$f(x,y,w,\frac{\partial^n w}{\partial x^n},\frac{\partial^n w}{\partial x^r \partial y^{n-r}},\ldots\ldots,\frac{\partial^n w}{\partial y^n},\frac{\partial^{n-1} w}{\partial x^{n-1}},\frac{\partial^{n-1} w}{\partial x^r \partial y^{n-r-1}},\ldots\ldots,\frac{\partial^{n-1} w}{\partial y^{n-1}},\frac{\partial^{n-2} w}{\partial x^{n-2}},\frac{\partial^{n-2} w}{\partial x^r \partial y^{n-r-2}}$$
$$,\ldots\ldots,\frac{\partial^{n-2} w}{\partial y^{n-2}},\ldots\ldots,\frac{\partial w}{\partial x},\frac{\partial w}{\partial y})=0; \qquad r=1,2,3,\ldots\ldots\ldots,n-1 \tag{1}$$

Where,

$w = w(x, y)$ is the unknown function of the dependent variables x, y .

Let us consider the dimensionless PDE over a rectangular r region $(a \times b) \in \Re$ in the short form:

$$(Lw)(\zeta,\eta)=0 \tag{2}$$

where L denotes the linear partial differential operator and $w = w(\zeta,\eta)$ is the dimensionless unknown function where

$$\zeta=\frac{x}{a} \text{ and } \eta=\frac{x}{b} .$$

Separation of variables assumes general solution of the form:

$$w(\zeta,\eta)=\sum_{m=1}^{M} g_m(\zeta) f_m(\eta) \tag{3}$$

Where $g_m(\zeta)$ and $f_m(\eta)$ are functions satisfying the boundary conditions of rectangular region at the boundaries $(\zeta = 0,1)$ and $(\eta = 0,1)$ respectively. Equation (3) is utilized to reduce Eq. (2) to the ordinary differential equation ODE:

$$\sum_{m=1}^{M}\sum_{n=0}^{N} C_{mn} f^{(n)}(\eta)=0 \tag{4}$$

where the function $f^{(n)}(\eta)$ is unknown function differentiated with order n while C_{mn} is integrated constant based on the known function $g_m(\zeta)$ so that:

$$C_{mn}=\int_0^1 g_m g_m^{(n)} d\zeta \tag{5}$$

Series solution of the reduced ODE (4) is:

$$w(\zeta,\eta)=\sum_{m=1}^{M} g_m(\zeta)\left[f_m(0)+\sum_{k=1}^{K} f_m^{(k)}(0)\frac{\eta^k}{k!}\right] \tag{6}$$

where

$$f_m^{(k)}(0)=\frac{d^k f}{d\eta^k} \quad \text{at} \quad \eta=0 \quad ;k=1,2,3,\ldots\ldots\ldots K \tag{7}$$

Consequently:

$$w(\zeta,\eta)=\sum_{m=1}^{M} g_m(\zeta)\ \Big\{\ f_m(0)+f'_m(0)\eta+f''_m(0)\frac{\eta^2}{2!} + f'''_m(0)\frac{\eta^3}{3!}+......+f_m^{(n-1)}(0)\frac{\eta^{n-1}}{(n-1)!}+f_m^{(n)}(0)\frac{\eta^n}{n!}+E_0\ \Big\} \tag{8}$$

Where,

$$E_0=\sum_{k=n}^{K} f_m^{(k)}(0)\frac{\eta^k}{k!} \tag{9}$$

The function E_0 is a truncated power series function, constructing on the constant coefficients $f_m^{(k)}(0);\ k=n,n+1,n+2,.......K$ which depend on the n initial values $f_m(0),\ f'_m(0),\ f''_m(0),\ f'''_m(0),\ ....,\ f_m^{(n-1)}(0)$ so that:

$$f_m^{(k)}(\eta)=\frac{d}{d\eta}f_m^{(k-1)}(\eta);\ \ k=n+1,n+2,n+3,.........K \tag{10}$$

where $f_m^{(n)}(\eta)$ is defined according to Eq. (4).

Initial values $f_m(0),\ f'_m(0),\ f''_m(0),\ f'''_m(0),\ ....,\ f_m^{(n-1)}(0)$ are accomplished from the known boundary condition at $(\eta=0,1)$.

III. SHAPE FUNCTION $g_m(\zeta)$ AND SEPARATION OF VARIABLES

Separation of variables is applied to create the shape function in only one variable of linear partial differential equations under boundary and initial conditions using

$$w(\zeta,\eta)=g(\zeta)\,f(\eta) \tag{11}$$

For cases which don't involve mixed derivatives such as the heat equation, Laplace equation, Helmholtz equation or wave equation, separation of variables is easily being done. But for cases involving mixed derivatives such as biharmonic equation, the PDE is not easily separated, but nonetheless Eq (11) may still be applied. According to separation, two single variable systems of ODE^S are obtained under boundary conditions. Satisfying the proposed boundary conditions, shape function is directly defined. For example, the dimensionless biharmonic equation is considered:

$$\nabla^2 w=0 \qquad \text{Where} \quad \nabla=\frac{\partial^2 w}{\partial\zeta^2}+\tau^2\frac{\partial^2 w}{\partial\eta^2} \tag{12}$$

where $\tau=\dfrac{a}{b}$ is the aspect ratio of the rectangular region.

Subsection from (11) in (12), gives:

$$\frac{g^{(4)}(\zeta)}{g(\zeta)}+2\tau^2\frac{g''(\zeta)}{g(x)}\frac{f''(\eta)}{f(\eta)}+\tau^4\frac{f^{(4)}(\eta)}{f(\eta)}=0 \tag{13}$$

Rewriting Eq. (13) in the form:

$$G(\zeta)+2\tau^2E(\zeta)H(\eta)+\tau^4F(\eta)=0 \tag{14}$$

Eliminating the first and last terms by differentiating Eq. (14) w.r.t ζ and η, yields:

$$E'(\zeta)H'(\eta)=0$$

This means that either $E(\zeta)$ or $H(\eta)$ must be a constant, say μ_1 or μ_2. Consequently,

$$-G(\zeta)=2\tau^2F(\zeta)H(\eta)+\tau^4F(\eta) \quad \text{or} \quad -\tau^4F(\eta)=G(\zeta)+2\tau^2F(\zeta)H(\eta) \text{ is constant.}$$

Differentiating all terms w.r.t ζ and η, one can prove that each of $G(\zeta)$ and $F(\eta)$ are constant, say ω_1 or ω_2. Thus two cases are obtained:

$$f^{(4)}(\eta)+2\mu_1\tau^2 f''(\eta)+\tau^4\omega_1 f(\eta)=0 \tag{15}$$

and

$$g^{(4)}(\zeta)+2\mu_2\tau^2 g''(\zeta)+\tau^4\omega_2 g(\zeta)=0 \tag{16}$$

Where

$$\left.\begin{array}{l} \dfrac{g''(\zeta)}{g(\zeta)}=\mu_1,\ \dfrac{g^{(4)}(\zeta)}{g(\zeta)}=\omega_1 \\ \dfrac{f''(\eta)}{f(\eta)}=\mu_2,\dfrac{f^{(4)}(\eta)}{f(\eta)}=\omega_2 \end{array}\right\} \tag{17}$$

Each case is homogenous ODE which can easily be solved by auxiliary equation. Because of $g^{(4)}(\zeta)=\omega_1 g(\zeta)$ and $g''(\zeta)=\mu_1 g(\zeta)$, then: $g^{(4)}(\zeta)=\mu_1 g''(\zeta)$. Thus, one can prove that $\omega_1=\mu_1^2$. Similarly $\omega_2=\mu_2^2$

The general solution of equation (16), is the shape function $g_m(\zeta)$ which is in the form:

$$g_m(\zeta)=A_1\sin(\alpha_m\zeta)+A_2\cos(\alpha_m\zeta)+A_3\sinh(\alpha_m\zeta+A_4\cosh(\alpha_m\zeta) \tag{18}$$

where α_m and A_1, A_2, A_3, A_4 are constant parameters depending on τ, μ_2 and boundary conditions at the edges of support at $(\zeta=0{,}1)$. Generalized techniques for separation of variables have been described by many researchers in literature [13], [14].

IV. APPLIED STUDY

A. Solution of PDE of Composite Plate

To illustrate and examine the present technique, an application of square composite plate is provided. The study will pay attention to the following fourth order PDE for vibration of a composite plate:

$$C_o w_{\zeta\zeta\zeta\zeta}+C_1\tau w_{\zeta\zeta\zeta\eta}+C_2\tau^2 w_{\zeta\zeta\eta\eta}+C_3\tau^3 w_{\zeta\eta\eta\eta}+C_4\tau^4 w_{\eta\eta\eta\eta}+\rho h a^4 w_{tt}=0 \tag{19}$$

where $w=w(\zeta,\eta,t)$ is the plate dimensionless displacement, the subscripts denote to partial derivatives with respect to the independent variables, ζ,η,t where ζ,η denote the dimensionless coordinates such as $\zeta=\frac{x}{a},\eta=\frac{y}{b}$ while t denotes time. The magnitude τ is the aspect ratio where $\tau=\frac{a}{b}$ and a,b are the dimensions of plate in Cartesian x,y directions respectively. The constant coefficients C_n, $n=0{,}1{,}2{,}3{,}4$ are composite plate parameters while ρ is plate mass per unit volume and h is the thickness of plate. The solution is assumed to be :

$$w(\zeta,\eta,t)=\sum_{m=1}^{M} g_m(\zeta)f_m(\eta)e^{i\omega t} \tag{20}$$

where $g_m(\zeta)$ is a known shape function satisfying the boundary conditions of plate at the two edges $(\zeta=0{,}1)$.

The assumed form (20) is used to reduce Eq. (19) to:

$$\begin{aligned}\sum_{m=1}^{M}[&\alpha_m C_4\tau^4 f_m''''(\eta)+\beta_m C_3\tau^3 f_m'''(\eta)+\gamma_m C_2\tau^2 f_m''(\eta)+p_m C_1\tau f_m'(\eta)\\ &+(q_m C_o-\omega^2\rho h a^4\alpha_m)f_m(\eta)]=0\end{aligned} \tag{21}$$

where

$$\alpha_{mn}=\int_0^1 g_m g_m d\zeta\ ,\ \beta_{mn}=\int_0^1 g_m g'_m\, d\zeta\ ,\ \gamma_{mn}=\int_0^1 g_m g''_m\, d\zeta\ ,$$
$$p_{mn}=\int_0^1 g_m g'''_m\, d\zeta\ ,\ q_{mn}=\int_0^1 g_m g''''_m\, d\zeta \tag{22}$$

and

$$f_m^{(4)}(\eta)=-\frac{\beta_m C_3}{\alpha_m \tau C_4}\, f'''_m(\eta)-\frac{\gamma_m C_2}{\alpha_m \tau^2 C_4}\, f''_m(\eta)-\frac{p_m C_1}{\alpha_m \tau^3 C_4}\, f'_m(\eta)$$
$$-\frac{(q_m \dfrac{C_o}{C_4}-\lambda^2\alpha_m)}{\alpha_m \tau^4}\, f_m(\eta) \tag{23}$$

The initial values $f_m(0),\ f'_m(0),\ f''_m(0),\ f'''_m(0)$ are defined due to the known boundary condition at $(\eta=0,1)$. Consequently, the derivatives $f_m^{(k)}(0);\ k=4,5,6,..........K$ are achieved according to:

$$f_m^{(k)}(\eta)=\frac{d}{d\eta}\, f_m^{(k-1)}(\eta);\ \ k=5,6,7,..........K \tag{24}$$

The final solution expressed in terms of the k^{th} power degree of η is:

$$\begin{aligned}
w(\zeta,\eta)=\textstyle\sum_{m=1}^{M} g_m(\zeta)\ \{\ & f_m(0)+f'_m(0)\eta+f''_m(0)\frac{\eta^2}{2!}\ +f'''_m(0)\frac{\eta^3}{3!}-\frac{1}{B_4}[B_3 f'''_m(0)\\
&+B_2 f''_m(0)+B_1 f'_m(0)\ \ +B_0 f_m(0)]\frac{\eta^4}{4!}-\frac{1}{B_4}[-\frac{B_3}{B_4}[B_3 f'''_m(0)+B_2 f''_m(0)\\
&+B_1 f'_m(0)+B_0 f_m(0)]+B_2 f'''_m(0)+B_1 f''_m(0)+B_0 f'_m(0)]\frac{\eta^5}{5!}\\
&-\frac{1}{B_4}[-\frac{B_3}{B_4}[-\frac{1}{B_4}[B_3 f'''_m(0)+B_2 f''_m(0)+B_1 f'_m(0)+B_0 f_m(0)]\ \ -\frac{B_2}{B_4}[B_3 f'''_m(0)\\
&+B_2 f''_m(0)+B_1 f'_m(0)+B_0 f_m(0)]\ \ \ +B_1 f'''_m(0)+B_0 f''_m(0)]\frac{\eta^6}{6!}+..........+f_m^{(k)}(0).\frac{\eta^k}{k!}\ \ \}
\end{aligned} \tag{25}$$

where

$B_0=(q_m \dfrac{C_o}{C_4}-\lambda_m^2\alpha_m)$, $\quad B_1=p_m C_1\tau$, $\quad B_2=\gamma_m C_2\tau^2$, $\quad B_3=\beta_m C_1\tau^3$, $\quad B_4=\alpha_m C_4\tau^4$ and

$\lambda_m^2=\omega^2 a^4\dfrac{\overline{m}}{C_4}$, $\overline{m}=\rho h$

B. Numerical Results

To realize the explicit and implicit solutions created from equation (25) under specified boundary conditions, the case of vibration of square simply supported SSSS orthotropic plate is examined here. The plate thickness is 0.75 mm, and the material properties and bending stiffness's of an orthotropic carbon-epoxy plate [15] are:

$D11=409.08$, $D22=23.28$, $D12=6.52$, $D16=D26=0$ and $D66=16.13$

The eigen values λ_{mn} of this case are, implicitly/ explicitly, obtained for truncation number $k=12$ of the power series. Three modes of vibrations are examined showing the following results:

1. ***First Mode*** $m=1$

The explicit eigen values $\lambda_{11},\lambda_{12},\lambda_{13},\lambda_{14},.........\lambda_{1n}$ when $m=1$ are obtained from the solution of the following algebraic equation:

$$1.809214146(10^{19}) - 2.015964353(10^{16})\lambda_1^{\ 2} + 8.920238451(10^{12})\lambda_1^{\ 4} - 2.206726296(10^{9})\lambda_1^{\ 6} + 2.623401997(10^{5})\lambda_1^{\ 8} = 0 \tag{26}$$

Thus, the first three values of λ_{1n} are

$\lambda_{11} = 44.7102107\ 1$, $\lambda_{12} = 56.2468947\ 3$, $\lambda_{13} = 49.6322492\ 1$

To illustrate the effectiveness of the present technique, the eigen function (mode shape) corresponding to an eigen value, say $\lambda_{11} = 44.7102107\ 1$, is expressed in explicit closed form:

$$\begin{aligned} w_{11}(\zeta,\eta) = {} & [1.0(10)^9\zeta + .5\zeta^2 - 1.646275758(10)^9\zeta^3 + .8002095248\zeta^4 + 8.134474111(10)^8\zeta^5 \\ & + 0.9113159054\zeta^6 - 1.911345212\ (10)^8\zeta^7 + 0.4493851800\zeta^8 + 2.630418326\ (10)^7\zeta^9 \\ & + 0.1478451609\zeta^{10} - 2.341320677\ (10)^6\zeta^{11}][\sin\pi\eta] = 0 \end{aligned} \tag{27}$$

This mode shape is visualized in Fig. 1.

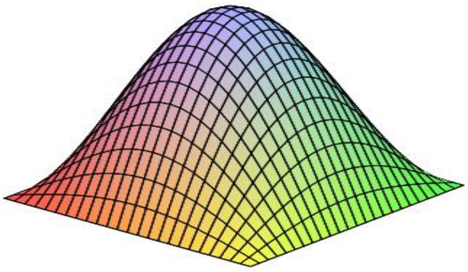

Fig. 1. Plate First Mode Shape, $\lambda_{11} = 44.7102107\ 1$ **,** $m = 1$**,** $n = 1$

2. *Second Mode* $m = 2$

Similarly, the explicit eigen value λ_{21}, λ_{22}, λ_{23}, λ_{24}, ...λ_{2n} are expressed in:

$$2.079942336(10)^{23} - 2.769390049(10)^{19}\lambda_2^{\ 2} + 1.388812471(10)^{15}\lambda_2^{\ 4} - 3.109683007(10)^{10}\lambda_2^{\ 6} + 2.62340201(10)^5\lambda_2^{\ 8} = 0 \tag{28}$$

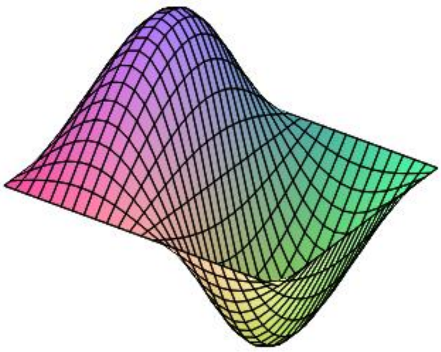

Fig. 2. Plate Second Mode Shape, $\lambda_{21} = 168.056354\ 79$ **,** $m = 2$**,** $n = 1$

Consequently, the first three values of λ_{2n} are

$\lambda_{21} = 168.056354\ 79$, $\lambda_{22} = 173.046490\ 3$, $\lambda_{23} = 175.633058\ 4$

Also, the mode shape of plate corresponding to:

$\lambda_{21} = 168.056354\ 79$ is illustrated in Fig. 2. and represented by the eigen function:

$$w_{21}(\zeta,\eta)=[1.0(10)^9\zeta+.5000000000\zeta^2 \quad -1.645483598(10)^9\zeta^3+3.200869939\zeta^4$$
$$+8.123348166(10)^8\zeta^5+9.385239634\ \zeta^6 \quad -1.908634427(10)^8\zeta^7+14.50546173\zeta^8 \tag{29}$$
$$+2.628349946(10)^7\zeta^9+13.97522067\ \zeta^{10} \quad -2.271220655(10)^6\ \zeta^{11}][\sin 2\pi\eta]=0$$

3. *Third Mode* $m=3$

The explicit eigen values $\lambda_{31},\lambda_{32},\lambda_{33},\lambda_{34},.........\lambda_{3n}$ are obtained from:

$$2.079942336(10)^{23}-2.769390049(10)^{19}\lambda_3{}^2 \quad +1.388812471(10)^{15}\lambda_3{}^4-3.109683007(10)^{10}\lambda_3{}^6 \tag{30}$$
$$+2.62340201(10)^5\lambda_3{}^8=0$$

In this case, the first three values of λ_{3n} are

λ_{31}= 374.769380 5 , λ_{32}=380.995864 6 , λ_{33}= 381.407368 2

Similarly, the eigen function corresponding to λ_{31}= 374.769380 5 is represented by the following equation and illustrated graphically in Fig. 3.

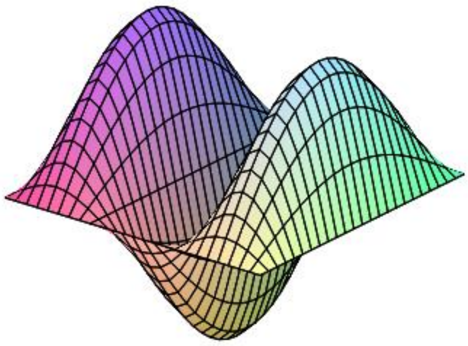

Fig. 3. Plate Third Mode Shape, λ_{31}= 374.769380 5 **,** $m=3$**,** $n=1$

$$w_{31}(x,y)=[1.0(10)^9\zeta+.5*\zeta^2-1.643310542(10)^9\zeta^3 \quad +7.204425902\zeta^4$$
$$+8.101302001(10)^8\zeta^5 \quad +44.02584561\zeta^6-1.902290394(10)^8\ \zeta^7$$
$$+143.6624973\zeta^8+2.593850618(10)^7\ \zeta^9+291.7429516\zeta^{10} \tag{31}$$
$$-2.510964708(10)^6\ \zeta^{11}][\sin 3\pi\eta]=0$$

V. CONCLUSUN

A new modified version of the method of separation of variables is presented. Considering mixed derivatives in the governing PDEs and direct substitutions of boundary conditions, mathematical manipulation becomes more difficult. Therefore publications of this type in literature are very few. The offered exact technique avoids the excessive effort needed to accomplish the alternative numerical solutions. The provided methodology reduces higher-order linear PDEs which incorporate mixed derivatives, into a set of easily to handle ODEs. The realization of the present technique provides simple, direct, and highly accurate closed form solutions for engineering problems governed by higher order linear PDE, involving mixed derivatives, under specified boundary conditions. The validity and reliability of the technique is examined through applying it to the case of vibration of orthotropic plate in three modes. The application provides the solutions in an accurate analytical simple fashion.

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