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Separable Fractional Differential Equations

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Abstract: **In this study, we use a new multiplication of fractional functions and chain rule for fractional derivatives to find the general solution of separable fractional differential equation, regarding the Jumarie type of modified Riemann-Liouville (R-L) fractional derivatives. On the other hand, an example is given for demonstrating the advantage of our result.**

Keywords: **New multiplication, Fractional functions, Chain rule, General solution, Separable fractional differential equations, Jumarie type of modified R-L fractional derivatives.**

I. INTRODUCTION

Fractional Calculus (FC) is a natural generalization of calculus that studies the possibility of computing derivatives and integrals of any real (or complex) order [1-3], i.e., not just of standard integer orders, such as first-derivative, second-derivative, etc. The history of FC started in 1695 when L'Hôpital raised the question as to the meaning of taking a fractional derivative such as $\frac{d^{1/2}y}{dx^{1/2}}$ and Leibniz replied [2]: "…This is an apparent paradox from which, one day, useful consequences will be drawn." Since then, eminent mathematicians such as Fourier, Abel, Liouville, Riemann, Weyl, Riesz, and many others contributed to the field, but until lately FC has played a negligible role in physics. We describe as a fractional equation an equation that contains fractional derivatives or integrals. Derivatives and integrals of fractional order have found many applications in recent studies in physics [2-3]. Broad classes of analytical methods have been proposed for solving fractional differential equations, such as the Adomian decomposition methods [4-5], variational iteration methods [6-10], differential transform methods [11] and the homotopy perturbation method [12-14].

Unlike standard calculus, there is no unique definition of derivative in FC. The definition of fractional derivative is given by many authors. The commonly used definitions are the Riemann-Liouvellie (R-L) fractional derivative [15], Caputo definition of fractional derivative (1967) [3], the Grunwald-Letinikov (G-L) fractional derivative [15], and Jumarie's modified R-L fractional derivative is used to avoid nonzero fractional derivative of a constant functions [16].

In this article, we can obtain the general solution of separable fractional differential equation, regarding the Jumarie type of modified R-L fractional derivatives. A new multiplication of fractional functions and chain rule are used and the main result we obtained is the generalization of general solution of separable ordinary differential equations. Furthermore, an example is proposed to demonstrate the advantage of our result.

II. PRELIMINARIES

Firstly, we introduce the fractional calculus adopted in this paper.

Definition 2.1: Suppose that α is a real number and m is a positive integer. The modified Riemann-Liouville fractional derivatives of Jumarie type ([16]) is defined by

$$ {}_aD_x^{\alpha}[f(x)] = \begin{cases} \frac{1}{\Gamma(-\alpha)}\int_a^x (x-\tau)^{-\alpha-1} f(\tau)d\tau, & \text{if } \alpha < 0 \\ \frac{1}{\Gamma(1-\alpha)}\frac{d}{dx}\int_a^x (x-\tau)^{-\alpha}[f(\tau)-f(a)]\,d\tau & \text{if } 0 \le \alpha < 1 \\ \frac{d^m}{dx^m}\left({}_aD_x^{\alpha-m}\right)[f(x)], & \text{if } m \le \alpha < m+1 \end{cases} \quad (1)$$

where $\Gamma(y) = \int_0^{\infty} t^{y-1}e^{-t}\,dt$ is the gamma function defined on $y > 0$. If $\left({}_aD_x^{\alpha}\right)^n[f(x)] = \left({}_aD_x^{\alpha}\right)\left({}_aD_x^{\alpha}\right)\cdots\left({}_aD_x^{\alpha}\right)[f(x)]$ exists, then $f(x)$ is called n-th order α-fractional differentiable function, and $\left({}_aD_x^{\alpha}\right)^n[f(x)]$ is the n-th order α-fractional derivative of $f(x)$. We note that $\left({}_aD_x^\alpha\right)^n \neq {}_aD_x^{n\alpha}$ in general. Moreover, we define the fractional integral of $f(x)$, ${}_aI_x^\alpha[f(x)] = {}_aD_x^{-\alpha}[f(x)]$, where $\alpha > 0$, and $f(x)$ is called α-integral function. We have the following property [17].

Proposition 2.2: *Suppose that α, β, c are real constants and $0 < \alpha \leq 1$, then*

$$ {}_0D_x^\alpha\left[x^\beta\right] = \frac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha+1)} x^{\beta-\alpha}, \text{ if } \beta \geq \alpha \tag{2}$$

$$ {}_0D_x^\alpha[c] = 0, \tag{3}$$

and

$$ \left({}_0I_x^\alpha\right)\left[x^\beta\right] = \frac{\Gamma(\beta+1)}{\Gamma(\beta+\alpha+1)} x^{\beta+\alpha}, \text{ if } \beta > -1. \tag{4}$$

Proposition 2.3 ([17])**:** *If $0 < \alpha \leq 1$ and $f(x)$ is a continuous function, then*

$$ \left({}_aD_x^\alpha\right)\left({}_aI_x^\alpha\right)[f(x)] = f(x). \tag{5}$$

Secondly, some fractional functions are introduced below.

Definition 2.4 ([18])**:** The Mittag-Leffler function is defined by

$$ E_\alpha(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(k\alpha+1)}, \tag{6}$$

where α is a real number, $\alpha > 0$, and z is a complex variable.

Definition 2.5 ([18])**:** $E_\alpha(\lambda x^\alpha)$ is called α-order fractional exponential function. The α-order fractional cosine and sine function are defined as follows:

$$ cos_\alpha(\lambda x^\alpha) = \sum_{k=0}^{\infty} \frac{(-1)^k \lambda^{2k} x^{2k\alpha}}{\Gamma(2k\alpha+1)}, \tag{7}$$

and

$$ sin_\alpha(\lambda x^\alpha) = \sum_{k=0}^{\infty} \frac{(-1)^k \lambda^{2k+1} x^{(2k+1)\alpha}}{\Gamma((2k+1)\alpha+1)}, \tag{8}$$

where $0 < \alpha \leq 1$, λ is a complex number, and x is a real variable.

The following is a new multiplication of fractional functions.

Definition 2.6 ([19])**:** Let λ, μ, z be complex numbers, $0 < \alpha \leq 1$, j, l, k be non-negative integers, and a_k, b_k be real numbers, $p_k(z) = \frac{1}{\Gamma(k\alpha+1)} z^k$ for all k. The $\otimes$ multiplication is defined by

$$ p_j(\lambda x^\alpha) \otimes p_l(\mu y^\alpha) = \frac{1}{\Gamma(j\alpha+1)}(\lambda x^\alpha)^j \otimes \frac{1}{\Gamma(l\alpha+1)}(\mu y^\alpha)^l = \frac{1}{\Gamma((j+l)\alpha+1)} \binom{j+l}{j} (\lambda x^\alpha)^j (\mu y^\alpha)^l, \tag{9}$$

where $\binom{j+l}{j} = \frac{(j+l)!}{j!l!}$.

If $f_\alpha(\lambda x^\alpha)$ and $g_\alpha(\mu y^\alpha)$ are two fractional functions,

$$ f_\alpha(\lambda x^\alpha) = \sum_{k=0}^{\infty} a_k\, p_k(\lambda x^\alpha) = \sum_{k=0}^{\infty} \frac{a_k}{\Gamma(k\alpha+1)} (\lambda x^\alpha)^k, \tag{10}$$

$$ g_\alpha(\mu y^\alpha) = \sum_{k=0}^{\infty} b_k\, p_k(\mu y^\alpha) = \sum_{k=0}^{\infty} \frac{b_k}{\Gamma(k\alpha+1)} (\mu y^\alpha)^k, \tag{11}$$

then we define

$$ f_\alpha(\lambda x^\alpha) \otimes g_\alpha(\mu y^\alpha) = \sum_{k=0}^{\infty} a_k\, p_k(\lambda x^\alpha) \otimes \sum_{k=0}^{\infty} b_k\, p_k(\mu y^\alpha)$$

$$ = \sum_{k=0}^{\infty}\left(\sum_{m=0}^{k} a_{k-m} b_m p_{k-m}(\lambda x^\alpha) \otimes p_m(\mu y^\alpha)\right). \tag{12}$$

Proposition 2.7: $f_\alpha(\lambda x^\alpha) \otimes g_\alpha(\mu y^\alpha) = \sum_{k=0}^{\infty} \frac{1}{\Gamma(k\alpha+1)} \sum_{m=0}^{k} \binom{k}{m} a_{k-m} b_m (\lambda x^\alpha)^{k-m} (\mu y^\alpha)^m$. (13)

Definition 2.8: Let $\left(f_\alpha(\lambda x^\alpha)\right)^{\otimes n} = f_\alpha(\lambda x^\alpha) \otimes \cdots \otimes f_\alpha(\lambda x^\alpha)$ be the n times product of the fractional function $f_\alpha(\lambda x^\alpha)$. If $f_\alpha(\lambda x^\alpha) \otimes g_\alpha(\lambda x^\alpha) = 1$, then $g_\alpha(\lambda x^\alpha)$ is called the $\otimes$ reciprocal of $f_\alpha(\lambda x^\alpha)$, and is denoted by $\left(f_\alpha(\lambda x^\alpha)\right)^{\otimes -1}$.

Remark 2.9: The $\otimes$ multiplication satisfies the commutative law and the associate law, and is the generalization of ordinary multiplication, since the $\otimes$ multiplication becomes the traditional multiplication if $\alpha = 1$.

Proposition 2.10: $E_\alpha(\lambda x^\alpha) \otimes E_\alpha(\mu y^\alpha) = E_\alpha(\lambda x^\alpha + \mu y^\alpha)$. (14)

Corollary 2.11: $E_\alpha(\lambda x^\alpha) \otimes E_\alpha(\mu x^\alpha) = E_\alpha((\lambda + \mu) x^\alpha)$. (15)

Definition 2.12: If $f(z) = \sum_{k=0}^{\infty} a_k z^k$, $g_\alpha(\mu x^\alpha) = \sum_{k=0}^{\infty} b_k p_k(\mu x^\alpha)$, then

$$f_{\otimes\alpha}\left(g_\alpha(\mu x^\alpha)\right) = \sum_{k=0}^{\infty} a_k \left(g_\alpha(\mu x^\alpha)\right)^{\otimes k}. \tag{16}$$

Theorem 2.13 (chain rule for fractional derivatives) ([19]): *If* $f(z) = \sum_{k=0}^{\infty} a_k z^k$, $g_\alpha(\mu x^\alpha) = \sum_{k=0}^{\infty} b_k p_k(\mu x^\alpha)$. *Let* $f_{\otimes\alpha}\left(g_\alpha(\mu x^\alpha)\right) = \sum_{k=0}^{\infty} a_k \left(g_\alpha(\mu x^\alpha)\right)^{\otimes k}$ *and* $f'_{\otimes\alpha}\left(g_\alpha(\mu x^\alpha)\right) = \sum_{k=1}^{\infty} a_k k \left(g_\alpha(\mu x^\alpha)\right)^{\otimes(k-1)}$, *then*

$$\left({}_0D_x^\alpha\right)\left[f_{\otimes\alpha}\left(g_\alpha(\mu x^\alpha)\right)\right] = f'_{\otimes\alpha}\left(g_\alpha(\mu x^\alpha)\right) \otimes \left({}_0D_x^\alpha\right)\left[g_\alpha(\mu x^\alpha)\right]. \tag{17}$$

Definition 2.14: Let x, y be real variables, $0 < \alpha \le 1$, $f_\alpha(x^\alpha)$, $g_\alpha(y^\alpha)$ be α-integral functions. Then

$${}_aD_x^\alpha[y(x^\alpha)] = f_\alpha(x^\alpha) \otimes g_\alpha(y^\alpha) \tag{18}$$

is called separable α-fractional differential equation.

III. MAIN RESULT

To obtain the general solution of Eq. (18), we need a lemma.

Lemma 3.1: *Suppose that* x, y *are real variables,* $0 < \alpha \le 1$, $h_\alpha(y^\alpha)$ *is* α*-integral function defined on* $[c, d]$, *and* $y = y(x^\alpha)$ *is* α*-differential function defined on* $[a, b]$. *Then*

$${}_aI_x^\alpha\left[h_\alpha(y^\alpha) \otimes {}_aD_x^\alpha[y(x^\alpha)]\right] = {}_cI_y^\alpha[h_\alpha(y^\alpha)] + C_1, \tag{19}$$

where C_1 is a constant.

Proof By chain rule for fractional derivatives, we have

$${}_aD_x^\alpha\left[{}_cI_y^\alpha[h_\alpha(y^\alpha)]\right] = h_\alpha(y^\alpha) \otimes {}_aD_x^\alpha[y(x^\alpha)]. \tag{20}$$

Therefore, the desired result holds. Q.e.d.

In the following, we obtain the general solution of Eq. (18).

Theorem 3.2: *Assume that* x, y *are real variables,* $0 < \alpha \le 1$, *and* $f_\alpha(x^\alpha)$, $g_\alpha(y^\alpha)^{\otimes -1}$ *are* α*-integral functions defined on* $[a, b], [c, d]$ *respectively. Then the separable* α*-fractional differential equation*

$${}_aD_x^\alpha[y(x^\alpha)] = f_\alpha(x^\alpha) \otimes g_\alpha(y^\alpha)$$

has the general solution

$${}_aI_x^\alpha[f(x^\alpha)] - {}_cI_y^\alpha\left[g_\alpha(y^\alpha)^{\otimes -1}\right] = C, \tag{21}$$

where C is a constant.

Proof Since

$$f_\alpha(x^\alpha) = g_\alpha(y^\alpha)^{\otimes -1} \otimes {}_aD_x^\alpha[y(x^\alpha)], \tag{22}$$

it follows that

$${}_aI_x^\alpha[f(x^\alpha)] = {}_aI_x^\alpha\left[g_\alpha(y^\alpha)^{\otimes -1} \otimes {}_aD_x^\alpha[y(x^\alpha)]\right]. \tag{23}$$

Using Lemma 3.1 yields

$$ {}_{a}I_{x}^{\alpha}[f(x^{\alpha})] = {}_{c}I_{y}^{\alpha}\left[g_{\alpha}(y^{\alpha})^{\otimes -1}\right] + C. \tag{24}$$

Thus, the desired result holds. Q.e.d.

IV. EXAMPLE

Next, we give an example to illustrate our result.

Example 4.1: Consider the separable $1/4$-fractional differential equation

$$ {}_{0}D_{x}^{1/4}\left[y\left(x^{1/4}\right)\right] = \left(E_{1/4}\left(x^{1/4}\right) + sin_{1/4}\left(x^{1/4}\right)\right) \otimes \left(y^{1/4} - cos_{1/4}\left(y^{1/4}\right) + E_{1/4}\left(y^{1/4}\right)\right)^{\otimes -1}, \tag{25}$$

where $y(0) = 0$.

Using Theorem 3.2 yields the general solution of Eq. (25)

$$ {}_{0}I_{x}^{1/4}\left[E_{1/4}\left(x^{1/4}\right) + sin_{1/4}\left(x^{1/4}\right)\right] - {}_{0}I_{y}^{1/4}\left[y^{1/4} - cos_{1/4}\left(y^{1/4}\right) + E_{1/4}\left(y^{1/4}\right)\right] = C. \tag{26}$$

That is,

$$ E_{1/4}\left(x^{1/4}\right) - cos_{1/4}\left(x^{1/4}\right) - \frac{\Gamma(5/4)}{\Gamma(3/2)}y^{1/2} + sin_{1/4}\left(y^{1/4}\right) - E_{1/4}\left(y^{1/4}\right) = C. \tag{27}$$

Since $y(0) = 0$, it follows that $C = -1$. Thus we get the particular solution of Eq. (25)

$$ E_{1/4}\left(x^{1/4}\right) - cos_{1/4}\left(x^{1/4}\right) - \frac{\Gamma(5/4)}{\Gamma(3/2)}y^{1/2} + sin_{1/4}\left(y^{1/4}\right) - E_{1/4}\left(y^{1/4}\right) = -1. \tag{28}$$

V. CONCLUSION

As mentioned, we mainly use a new multiplication of fractional functions and chain rule for fractional derivatives to find the general solution of separable fractional differential equations. In the future, we also use the Jumarie type of modified R-L fractional derivatives and the new multiplication to extend the research topics to the problems of engineering mathematics and fractional calculus.

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