International Journal of Interdisciplinary Research and Innovations ISSN 2348-1218 (print)
ISSN 2348-1226 (online)
Vol. 8, Issue 4, pp: (100-105), Month: October - December 2020, Available at: www.researchpublish.com

A Study of Exact Fractional Differential Equations

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Abstract: **In this article, a new multiplication of fractional functions and chain rule for fractional derivatives are used to obtain the general solution of exact fractional differential equation (FDE), regarding the Jumarie type of modified Riemann-Liouville (R-L) fractional derivatives. Furthermore, an example is given for demonstrating the advantage of our result.**

Keywords: **New multiplication, Fractional functions, Chain rule, Exact FDE, Jumarie type of modified R-L fractional derivatives.**

I. INTRODUCTION

Fractional differential equations have excited in recent years a considerable interest both in mathematics and in applications. They were used in modeling of many physical and chemical processes and in engineering [1-4]. In its turn, mathematical aspects of fractional differential equations and methods of their solution were discussed by many authors: the iteration method in [5], the series method in [1], the Fourier transform technique in [6-7], special methods for fractional differential equations of rational order or for equations of special type in [8-13], the operational calculus method in [14-15].

Unlike standard calculus, there is no unique definition of derivative in fractional calculus. The definition of fractional derivative is given by many authors. The commonly used definitions are the Riemann-Liouvellie (R-L) fractional derivative [16], Caputo definition of fractional derivative [17], the Grunwald-Letinikov (G-L) fractional derivative [16], and Jumarie's modified R-L fractional derivative is used to avoid nonzero fractional derivative of a constant functions [18]. After the first congress at the University of New Haven, in 1974, fractional calculus has developed and several applications emerged in many areas of scientific knowledge. As a consequence, distinct approaches to solve problems involving the derivative were proposed and distinct definitions of the fractional derivative are available in the literature.

In this paper, the general solution of exact fractional differential equation (FDE), regarding the Jumarie type of modified R-L fractional derivatives can be obtained by using a new multiplication of fractional functions and chain rule for fractional derivatives. In fact, the result we obtained is the generalization of general solution of exact ordinary differential equations. On the other hand, an example is proposed to demonstrate the advantage of our result.

II. PRELIMINARIES

In the following, fractional calculus used in this paper is introduced.

Definition 2.1: If α is a real number and m is a positive integer. Then we define the modified Riemann-Liouville fractional derivatives of Jumarie type ([16])

$$
{}_aD_x^{\alpha}[f(x)] = \begin{cases} \frac{1}{\Gamma(-\alpha)}\int_a^x (x-\tau)^{-\alpha-1} f(\tau)d\tau, & \text{if } \alpha < 0 \\ \frac{1}{\Gamma(1-\alpha)}\frac{d}{dx}\int_a^x (x-\tau)^{-\alpha}[f(\tau)-f(a)]\,d\tau & \text{if } 0 \le \alpha < 1 \\ \frac{d^m}{dx^m}\left({}_aD_x^{\alpha-m}\right)[f(x)], & \text{if } m \le \alpha < m+1 \end{cases} \quad (1)
$$

where $\Gamma(y)=\int_0^{\infty} t^{y-1}e^{-t}dt$ is the gamma function defined on $y>0$. If $\left({}_aD_x^{\alpha}\right)^n[f(x)]=\left({}_aD_x^{\alpha}\right)\left({}_aD_x^{\alpha}\right)\cdots\left({}_aD_x^{\alpha}\right)[f(x)]$ exists, then $f(x)$ is called n-th order α-fractional differentiable function, and $\left({}_aD_x^{\alpha}\right)^n[f(x)]$ is the n-th order α-fractional derivative of $f(x)$. Moreover, we define the α-fractional integral of $f(x)$, ${}_aI_x^{\alpha}[f(x)]={}_aD_x^{-\alpha}[f(x)]$, where $\alpha>0$, and $f(x)$ is called α-fractional integral function. Furthermore, if $M_{\alpha}(x^{\alpha},y^{\alpha})$ is a two-variable α-fractional function defined on $[a,b]\times[c,d]$, then we define ${}_a\partial_x^{\alpha}[M_{\alpha}(x^{\alpha},y^{\alpha})]$ and ${}_c\partial_y^{\alpha}[M_{\alpha}(x^{\alpha},y^{\alpha})]$ are α-fractional partial derivatives with respect to x and y respectively. And, ${}_aJ_x^{\alpha}[M_{\alpha}(x^{\alpha},y^{\alpha})]$ and ${}_cJ_y^{\alpha}[M_{\alpha}(x^{\alpha},y^{\alpha})]$ are α-fractional integrals with respect to x and y respectively.

Proposition 2.2: *Suppose that α,β,c are real constants and $0<\alpha\leq 1$, then*

$$ {}_0D_x^{\alpha}[x^{\beta}]=\frac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha+1)}x^{\beta-\alpha},\ if\ \beta\geq\alpha \tag{2}$$

$$ {}_0D_x^{\alpha}[c]=0, \tag{3}$$

and

$$\left({}_0I_x^{\alpha}\right)[x^{\beta}]=\frac{\Gamma(\beta+1)}{\Gamma(\beta+\alpha+1)}x^{\beta+\alpha},\ \ if\ \beta>-1. \tag{4}$$

In the following, we define a new multiplication of fractional functions.

Definition 2.3 ([21])**:** Let λ,μ,z be complex numbers, $0<\alpha\leq 1$, j,l,k be non-negative integers, and a_k,b_k be real numbers, $p_k(z)=\frac{1}{\Gamma(k\alpha+1)}z^k$ for all k. The $\otimes$ multiplication is defined by

$$p_j(\lambda x^{\alpha})\otimes p_l(\mu y^{\alpha})=\frac{1}{\Gamma(j\alpha+1)}(\lambda x^{\alpha})^j\otimes\frac{1}{\Gamma(l\alpha+1)}(\mu y^{\alpha})^l=\frac{1}{\Gamma((j+l)\alpha+1)}\binom{j+l}{j}(\lambda x^{\alpha})^j(\mu y^{\alpha})^l, \tag{5}$$

where $\binom{j+l}{j}=\frac{(j+l)!}{j!\,l!}$.

If $f_{\alpha}(\lambda x^{\alpha})$ and $g_{\alpha}(\mu y^{\alpha})$ are two fractional functions,

$$f_{\alpha}(\lambda x^{\alpha})=\sum_{k=0}^{\infty}a_k\,p_k(\lambda x^{\alpha})=\sum_{k=0}^{\infty}\frac{a_k}{\Gamma(k\alpha+1)}(\lambda x^{\alpha})^k, \tag{6}$$

$$g_{\alpha}(\mu y^{\alpha})=\sum_{k=0}^{\infty}b_k\,p_k(\mu y^{\alpha})=\sum_{k=0}^{\infty}\frac{b_k}{\Gamma(k\alpha+1)}(\mu y^{\alpha})^k, \tag{7}$$

then we define

$$\begin{aligned} f_{\alpha}(\lambda x^{\alpha})\otimes g_{\alpha}(\mu y^{\alpha}) &= \sum_{k=0}^{\infty}a_k\,p_k(\lambda x^{\alpha})\otimes\sum_{k=0}^{\infty}b_k\,p_k(\mu y^{\alpha}) \\ &=\sum_{k=0}^{\infty}\left(\sum_{m=0}^{k}a_{k-m}b_m p_{k-m}(\lambda x^{\alpha})\otimes p_m(\mu y^{\alpha})\right). \end{aligned} \tag{8}$$

Proposition 2.4: $$f_{\alpha}(\lambda x^{\alpha})\otimes g_{\alpha}(\mu y^{\alpha})=\sum_{k=0}^{\infty}\frac{1}{\Gamma(k\alpha+1)}\sum_{m=0}^{k}\binom{k}{m}a_{k-m}b_m\,(\lambda x^{\alpha})^{k-m}(\mu y^{\alpha})^m. \tag{9}$$

Definition 2.5: Let $\left(f_{\alpha}(\lambda x^{\alpha})\right)^{\otimes n}=f_{\alpha}(\lambda x^{\alpha})\otimes\cdots\otimes f_{\alpha}(\lambda x^{\alpha})$ be the n times product of the fractional function $f_{\alpha}(\lambda x^{\alpha})$. If $f_{\alpha}(\lambda x^{\alpha})\otimes g_{\alpha}(\lambda x^{\alpha})=1$, then $g_{\alpha}(\lambda x^{\alpha})$ is called the $\otimes$ reciprocal of $f_{\alpha}(\lambda x^{\alpha})$, and is denoted by $\left(f_{\alpha}(\lambda x^{\alpha})\right)^{\otimes -1}$.

Remark 2.6: The $\otimes$ multiplication satisfies the commutative law and the associate law, and is the generalization of ordinary multiplication, since the $\otimes$ multiplication becomes the traditional multiplication if $\alpha=1$.

Definition 2.7: If $f(z)=\sum_{k=0}^{\infty}a_k\,z^k$, $g_{\alpha}(\mu x^{\alpha})=\sum_{k=0}^{\infty}b_k\,p_k(\mu x^{\alpha})$, then

$$f_{\otimes\alpha}\left(g_{\alpha}(\mu x^{\alpha})\right)=\sum_{k=0}^{\infty}a_k\left(g_{\alpha}(\mu x^{\alpha})\right)^{\otimes k}. \tag{10}$$

Theorem 2.8 (chain rule for fractional derivatives) ([21])**:** *If $f(z)=\sum_{k=0}^{\infty}a_k\,z^k$, $g_{\alpha}(\mu x^{\alpha})=\sum_{k=0}^{\infty}b_k\,p_k(\mu x^{\alpha})$. Let $f_{\otimes\alpha}\left(g_{\alpha}(\mu x^{\alpha})\right)=\sum_{k=0}^{\infty}a_k\left(g_{\alpha}(\mu x^{\alpha})\right)^{\otimes k}$ and $f'_{\otimes\alpha}\left(g_{\alpha}(\mu x^{\alpha})\right)=\sum_{k=1}^{\infty}a_k\,k\left(g_{\alpha}(\mu x^{\alpha})\right)^{\otimes(k-1)}$, then*

$$\left({}_0D_x^{\alpha}\right)\left[f_{\otimes\alpha}\left(g_{\alpha}(\mu x^{\alpha})\right)\right]=f'_{\otimes\alpha}\left(g_{\alpha}(\mu x^{\alpha})\right)\otimes\left({}_0D_x^{\alpha}\right)[g_{\alpha}(\mu x^{\alpha})]. \tag{11}$$

Definition 2.9: If x, y are real variables, $y = y(x): [a, b] \to [c, d]$, $y(a) = c$, $0 < \alpha \leq 1$. Suppose that $M_\alpha(x^\alpha, y^\alpha)$, $N_\alpha(x^\alpha, y^\alpha)$ defined on $[a, b] \times [c, d]$, and have continuous first-order α-fractional partial derivatives, $N_\alpha(x^\alpha, y^\alpha) \neq 0$, then

$$ {}_aD_x^\alpha[y(x^\alpha)] = -M_\alpha(x^\alpha, y^\alpha) \otimes \left(N_\alpha(x^\alpha, y^\alpha)\right)^{\otimes -1} \tag{12} $$

is called exact α-fractional differential equation, if ${}_c\partial_y^\alpha[M_\alpha(x^\alpha, y^\alpha)] = {}_a\partial_x^\alpha[N_\alpha(x^\alpha, y^\alpha)]$.

III. MAJOR RESULT

To obtain the main result, a lemma is needed.

Lemma 3.1: *Suppose that the assumptions are the same as Definition 2.9, and* C_1 *is a constant. If* ${}_c\partial_y^\alpha[M_\alpha(x^\alpha, y^\alpha)] = {}_a\partial_x^\alpha[N_\alpha(x^\alpha, y^\alpha)]$, *then*

$$ {}_aJ_x^\alpha[M_\alpha(x^\alpha, y^\alpha)] + {}_cJ_y^\alpha\left[N_\alpha(x^\alpha, y^\alpha) - {}_c\partial_y^\alpha\left[{}_aJ_x^\alpha[M_\alpha(x^\alpha, y^\alpha)]\right]\right] $$

$$ = {}_cJ_y^\alpha[N_\alpha(x^\alpha, y^\alpha)] + {}_aJ_x^\alpha\left[M_\alpha(x^\alpha, y^\alpha) - {}_a\partial_x^\alpha\left[{}_cJ_y^\alpha[N_\alpha(x^\alpha, y^\alpha)]\right]\right] + C_1 \tag{13} $$

Proof Let $u(x^\alpha, y^\alpha) = {}_aJ_x^\alpha[M_\alpha(x^\alpha, y^\alpha)] + {}_cJ_y^\alpha\left[N_\alpha(x^\alpha, y^\alpha) - {}_c\partial_y^\alpha\left[{}_aJ_x^\alpha[M_\alpha(x^\alpha, y^\alpha)]\right]\right]$,

and

$$ v(x^\alpha, y^\alpha) = {}_cJ_y^\alpha[N_\alpha(x^\alpha, y^\alpha)] + {}_aJ_x^\alpha\left[M_\alpha(x^\alpha, y^\alpha) - {}_a\partial_x^\alpha\left[{}_cJ_y^\alpha[N_\alpha(x^\alpha, y^\alpha)]\right]\right]. $$

Since

$$ {}_a\partial_x^\alpha\left[N_\alpha(x^\alpha, y^\alpha) - {}_c\partial_y^\alpha\left[{}_aJ_x^\alpha[M_\alpha(x^\alpha, y^\alpha)]\right]\right] $$

$$ = {}_a\partial_x^\alpha[N_\alpha(x^\alpha, y^\alpha)] - {}_a\partial_x^\alpha\left[{}_c\partial_y^\alpha[J_x^\alpha[M_\alpha(x^\alpha, y^\alpha)]]\right] $$

$$ = {}_a\partial_x^\alpha[N_\alpha(x^\alpha, y^\alpha)] - {}_c\partial_y^\alpha\left[{}_a\partial_x^\alpha\left[{}_aJ_x^\alpha[M_\alpha(x^\alpha, y^\alpha)]\right]\right] $$

$$ = {}_a\partial_x^\alpha[N_\alpha(x^\alpha, y^\alpha)] - {}_c\partial_y^\alpha[M_\alpha(x^\alpha, y^\alpha)] $$

$$ = 0. $$

It follows that

$$ N_\alpha(x^\alpha, y^\alpha) - {}_c\partial_y^\alpha\left[{}_aJ_x^\alpha[M_\alpha(x^\alpha, y^\alpha)]\right] = \varphi(y^\alpha), \tag{14} $$

for some α-fractional function $\varphi(y^\alpha)$.

And hence,

$$ {}_a\partial_x^\alpha[u(x^\alpha, y^\alpha)] = M_\alpha(x^\alpha, y^\alpha). \tag{15} $$

Similarly,

$$ {}_c\partial_y^\alpha[v(x^\alpha, y^\alpha)] = N_\alpha(x^\alpha, y^\alpha). \tag{16} $$

Using ${}_c\partial_y^\alpha[M_\alpha(x^\alpha, y^\alpha)] = {}_a\partial_x^\alpha[N_\alpha(x^\alpha, y^\alpha)]$ yields

$$ {}_c\partial_y^\alpha\left[{}_a\partial_x^\alpha[u(x^\alpha, y^\alpha)]\right] = {}_a\partial_x^\alpha\left[{}_c\partial_y^\alpha[v(x^\alpha, y^\alpha)]\right] = {}_c\partial_y^\alpha\left[{}_a\partial_x^\alpha[v(x^\alpha, y^\alpha)]\right]. \tag{17} $$

Therefore, Eq. (13) holds.

In the following, we obtain the general solution of Eq. (12).

Theorem 3.2: *Let the assumptions be the same as Definition 2.13, and C be a constant. If the α-fractional differential equation*

$$ {}_aD_x^{\alpha}[y(x^{\alpha})] = -M_{\alpha}(x^{\alpha},y^{\alpha}) \otimes \left(N_{\alpha}(x^{\alpha},y^{\alpha})\right)^{\otimes -1} $$

is exact, then it has the general solution

$$ {}_aJ_x^{\alpha}[M_{\alpha}(x^{\alpha},y^{\alpha})] + {}_cJ_y^{\alpha}\left[N_{\alpha}(x^{\alpha},y^{\alpha}) - {}_c\partial_y^{\alpha}\left[{}_aJ_x^{\alpha}[M_{\alpha}(x^{\alpha},y^{\alpha})]\right]\right] = C. \tag{18} $$

Proof Let $u(x^{\alpha},y^{\alpha}) = {}_aJ_x^{\alpha}[M_{\alpha}(x^{\alpha},y^{\alpha})] + {}_cJ_y^{\alpha}\left[N_{\alpha}(x^{\alpha},y^{\alpha}) - {}_c\partial_y^{\alpha}\left[{}_aJ_x^{\alpha}[M_{\alpha}(x^{\alpha},y^{\alpha})]\right]\right]$. By Lemma 3.1, we have

$$ {}_a\partial_x^{\alpha}[u(x^{\alpha},y^{\alpha})] = M_{\alpha}(x^{\alpha},y^{\alpha}),\ {}_c\partial_y^{\alpha}[u(x^{\alpha},y^{\alpha})] = N_{\alpha}(x^{\alpha},y^{\alpha}). $$

On the other hand, using chain rule for fractional derivatives yields

$$ {}_aD_x^{\alpha}[u(x^{\alpha},y^{\alpha})] = {}_a\partial_x^{\alpha}[u(x^{\alpha},y^{\alpha})] + {}_c\partial_y^{\alpha}[u(x^{\alpha},y^{\alpha})] \otimes {}_aD_x^{\alpha}[y(x^{\alpha})]. \tag{19} $$

Thus,

$$ \begin{aligned} {}_aD_x^{\alpha}[u(x^{\alpha},y^{\alpha})] &= M_{\alpha}(x^{\alpha},y^{\alpha}) + N_{\alpha}(x^{\alpha},y^{\alpha}) \otimes {}_aD_x^{\alpha}[y(x^{\alpha})] \\ &= M_{\alpha}(x^{\alpha},y^{\alpha}) + N_{\alpha}(x^{\alpha},y^{\alpha}) \otimes -M_{\alpha}(x^{\alpha},y^{\alpha}) \otimes \left(N_{\alpha}(x^{\alpha},y^{\alpha})\right)^{\otimes -1} \\ &= 0. \end{aligned} $$

And hence, $u(x^{\alpha},y^{\alpha}) = C$. Q.e.d.

IV. EXAMPLE

In the following, we give an example to illustrate our result.

Example 4.1: Consider the $1/3$-fractional differential equation

$$ {}_0D_x^{1/3}\left[y\left(x^{1/3}\right)\right] = -\left(3x^{2/3} + 6x^{1/3}y^{2/3}\right) \otimes \left(6x^{2/3}y^{1/3} + 4y\right)^{\otimes -1}. \tag{20} $$

Since

$$ {}_0\partial_y^{1/3}\left[3x^{2/3} + 6x^{1/3}y^{2/3}\right] = \frac{6\Gamma(5/3)}{\Gamma(4/3)} x^{1/3}y^{1/3}, \tag{21} $$

and

$$ {}_0\partial_x^{1/3}\left[6x^{2/3}y^{1/3} + 4y\right] = \frac{6\Gamma(5/3)}{\Gamma(4/3)} x^{1/3}y^{1/3}. \tag{22} $$

It follows from Definition 2.9 that Eq. (20) is exact. Suppose that $y(0) = 0$, then by Theorem 3.2, the general solution of Eq. (20) is

$$ {}_0J_x^{1/3}\left[3x^{2/3} + 6x^{1/3}y^{2/3}\right] + {}_0J_y^{1/3}\left[6x^{2/3}y^{1/3} + 4y - {}_0\partial_y^{1/3}\left[{}_0J_x^{1/3}\left[3x^{2/3} + 6x^{1/3}y^{2/3}\right]\right]\right] = C. \tag{23} $$

Therefore,

$$ \frac{3\Gamma\left(\Gamma(5/3)\right)}{\Gamma(2)} x + \frac{6\Gamma\left(\Gamma(4/3)\right)}{\Gamma(5/3)} x^{2/3}y^{2/3} + \frac{4\Gamma(2)}{\Gamma(7/3)} y^{4/3} = C. \tag{24} $$

Since $y(0) = 0$, it follows that $C = 0$, and hence we get the particular solution

$$ \frac{3\Gamma\left(\Gamma(5/3)\right)}{\Gamma(2)} x + \frac{6\Gamma\left(\Gamma(4/3)\right)}{\Gamma(5/3)} x^{2/3}y^{2/3} + \frac{4\Gamma(2)}{\Gamma(7/3)} y^{4/3} = 0\,. \tag{25} $$

V. CONCLUSION

In this paper, we use a new multiplication of fractional functions and chain rule for fractional derivatives to find the general solution of exact fractional differential equations. In fact, the new multiplication we defined is a natural operation in fractional calculus, and the result we obtained is a generalization of general solution of exact ordinary differential equations. In the future, we will use the modified R-L fractional derivatives of Jumarie type and the new multiplication to extend the research topics to the problems of fractional calculus and engineering mathematics.

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