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# Research on First Order Linear Fractional Differential Equations

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*Abstract:* **In this paper, we use product rule of fractional functions, integrating factor, and constant variation method to obtain the general solution of first order linear fractional differential equation (LFDE), regarding Jumarie's modified Riemann-Liouville (R-L) fractional derivative. Moreover, an example is proposed for demonstrating the advantage of our result.**

*Keywords:* **product rule, integrating factor, constant variation method, first order LFDE, Jumarie's modified R-L fractional derivative.**

## I. INTRODUCTION

The classical calculus provides a power tool to model and explain many important dynamically processes in most parts of applied areas of the sciences. But There are many complex systems in nature with anomalous dynamics, including biology, chemistry, physics, geology, astrophysics and social sciences, and more in particular in transport of chemical contaminant through water around rocks, dynamics of viscoelastic materials as polymers, signals theory, control theory, electromagnetic theory, and many more their dynamics cannot be characterized by classical derivative models. (for detail [1-6]).

Fractional calculus is the calculus of differentiation and integration of non-integer orders. During last three decades or so, fractional calculus has gained much attention due to its demonstrated applications in various fields of science and engineering [6-9]. There are many good textbooks of fractional calculus and fractional differential equations, such as [10-12]. For various applications of fractional calculus in physics, see [3, 6, 7, 8, 9]. Unlike standard calculus, there is no unique definition of derivation and integration in fractional calculus. The commonly used definition is the Riemann-Liouville (R-L) fractional derivative [5]. Other useful definitions include Caputo definition of fractional derivative (1967) [13], the Grunwald-Letinikov (G-L) fractional derivative [5], and Jumarie's modified R-L fractional derivative is used to avoid nonzero fractional derivative of constant functions [14].

In this paper, the first order linear fractional differential equation (LFDE), regarding the Jumarie type of modified R-L fractional derivatives is the generalization of first order linear ordinary differential equation. We define a new multiplication of fractional functions and use the product rule and the integrating factor method to obtain the general solution of the first order LFDE. On the other hand, an example is given to demonstrate the advantage of our result.

## II. MATERIALS AND METHODS

At first, the fractional calculus adopted in this paper is introduced below.

**Definition 2.1:** Suppose that $\alpha$ is a real number and $m$ is a positive integer. The modified Riemann-Liouville fractional derivatives of Jumarie type ([15]) is defined by

$$
{}_aD_x^{\alpha}[f(x)] = \begin{cases} \frac{1}{\Gamma(-\alpha)}\int_a^x (x-\tau)^{-\alpha-1} f(\tau)d\tau, & \text{if } \alpha < 0 \\ \frac{1}{\Gamma(1-\alpha)}\frac{d}{dx}\int_a^x (x-\tau)^{-\alpha}[f(\tau)-f(a)]\,d\tau & \text{if } 0 \le \alpha < 1 \\ \frac{d^m}{dx^m}\left({}_aD_x^{\alpha-m}\right)[f(x)], & \text{if } m \le \alpha < m+1 \end{cases} \tag{1}
$$

where $\Gamma(y)=\int_0^{\infty} t^{y-1}e^{-t}\,dt$ is the gamma function defined on $y>0$. If $\left({}_aD_x^{\alpha}\right)^n[f(x)]=\left({}_aD_x^{\alpha}\right)\left({}_aD_x^{\alpha}\right)\cdots\left({}_aD_x^{\alpha}\right)[f(x)]$ exists, then $f(x)$ is called $n$-th order $\alpha$-fractional differentiable function, and $\left({}_aD_x^{\alpha}\right)^n[f(x)]$ is the $n$-th order $\alpha$-fractional derivative of $f(x)$. We note that $\left({}_aD_x^{\alpha}\right)^n \neq {}_aD_x^{n\alpha}$ in general. On the other hand, we define the fractional integral of $f(x)$, ${}_aI_x^{\alpha}[f(x)]={}_aD_x^{-\alpha}[f(x)]$, where $\alpha>0$, and $f(x)$ is called α-integral function. We have the following property [16].

**Proposition 2.2:** *Let $\alpha,\beta,c$ be real numbers and $\beta\geq\alpha>0$, then*

$$ {}_0D_x^{\alpha}\left[x^{\beta}\right]=\frac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha+1)}x^{\beta-\alpha}, \tag{2}$$

*and*

$$ {}_0D_x^{\alpha}[c]=0. \tag{3}$$

fSecondly, we introduce some fractional functions.

**Definition 2.3** ([17])**:** The Mittag-Leffler function is defined by

$$E_{\alpha}(z)=\sum_{k=0}^{\infty}\frac{z^k}{\Gamma(k\alpha+1)}, \tag{4}$$

where $\alpha$ is a real number, $\alpha>0$, and $z$ is a complex variable.

**Definition 2.4** ([18])**:** $E_{\alpha}(\lambda x^{\alpha})$ is called $\alpha$-order fractional exponential function. The $\alpha$-order fractional cosine and sine function are defined as follows:

$$cos_{\alpha}(\lambda x^{\alpha})=\sum_{k=0}^{\infty}\frac{(-1)^k\lambda^{2k}x^{2k\alpha}}{\Gamma(2k\alpha+1)}, \tag{5}$$

and

$$sin_{\alpha}(\lambda x^{\alpha})=\sum_{k=0}^{\infty}\frac{(-1)^k\lambda^{2k+1}x^{(2k+1)\alpha}}{\Gamma((2k+1)\alpha+1)}, \tag{6}$$

where $0<\alpha\leq 1$, $\lambda$ is a complex number, and $x$ is a real variable.

The following is a new multiplication of fractional functions.

**Definition 2.5** ([19])**:** Let $\lambda,\mu,z$ be complex numbers, $0<\alpha\leq 1$, $j,l,k$ be non-negative integers, and $a_k,b_k$ be real numbers, $p_k(z)=\frac{1}{\Gamma(k\alpha+1)}z^k$ for all $k$. The $\otimes$ multiplication is defined by

$$p_j(\lambda x^{\alpha})\otimes p_l(\mu y^{\alpha})=\frac{1}{\Gamma(j\alpha+1)}(\lambda x^{\alpha})^j\otimes\frac{1}{\Gamma(l\alpha+1)}(\mu y^{\alpha})^l=\frac{1}{\Gamma((j+l)\alpha+1)}\binom{j+l}{j}(\lambda x^{\alpha})^j(\mu y^{\alpha})^l, \tag{7}$$

where $\binom{j+l}{j}=\frac{(j+l)!}{j!l!}$.

If $f_{\alpha}(\lambda x^{\alpha})$ and $g_{\alpha}(\mu y^{\alpha})$ are two fractional functions,

$$f_{\alpha}(\lambda x^{\alpha})=\sum_{k=0}^{\infty}a_k\,p_k(\lambda x^{\alpha})=\sum_{k=0}^{\infty}\frac{a_k}{\Gamma(k\alpha+1)}(\lambda x^{\alpha})^k, \tag{8}$$

$$g_{\alpha}(\mu y^{\alpha})=\sum_{k=0}^{\infty}b_k\,p_k(\mu y^{\alpha})=\sum_{k=0}^{\infty}\frac{b_k}{\Gamma(k\alpha+1)}(\mu y^{\alpha})^k, \tag{9}$$

then we define

$$f_{\alpha}(\lambda x^{\alpha})\otimes g_{\alpha}(\mu y^{\alpha})=\sum_{k=0}^{\infty}a_k\,p_k(\lambda x^{\alpha})\otimes\sum_{k=0}^{\infty}b_k\,p_k(\mu y^{\alpha})$$

$$=\sum_{k=0}^{\infty}\left(\sum_{m=0}^{k}a_{k-m}b_m p_{k-m}(\lambda x^{\alpha})\otimes p_m(\mu y^{\alpha})\right). \tag{10}$$

**Proposition 2.6:** $$f_{\alpha}(\lambda x^{\alpha})\otimes g_{\alpha}(\mu y^{\alpha})=\sum_{k=0}^{\infty}\frac{1}{\Gamma(k\alpha+1)}\sum_{m=0}^{k}\binom{k}{m}a_{k-m}b_m\,(\lambda x^{\alpha})^{k-m}(\mu y^{\alpha})^m. \tag{11}$$

**Definition 2.7:** Let $\left(f_{\alpha}(\lambda x^{\alpha})\right)^{\otimes n}=f_{\alpha}(\lambda x^{\alpha})\otimes\cdots\otimes f_{\alpha}(\lambda x^{\alpha})$ be the $n$ times product of the fractional function $f_{\alpha}(\lambda x^{\alpha})$. If $f_{\alpha}(\lambda x^{\alpha})\otimes g_{\alpha}(\lambda x^{\alpha})=1$, then $g_{\alpha}(\lambda x^{\alpha})$ is called the $\otimes$ reciprocal of $f_{\alpha}(\lambda x^{\alpha})$, and is denoted by $\left(f_{\alpha}(\lambda x^{\alpha})\right)^{\otimes -1}$.

**Remark 2.8:** The $\otimes$ multiplication satisfies the commutative law and the associate law, and is the generalization of ordinary multiplication, since the $\otimes$ multiplication becomes the traditional multiplication if $\alpha = 1$.

**Proposition 2.9:** $E_\alpha(\lambda x^\alpha) \otimes E_\alpha(\mu y^\alpha) = E_\alpha(\lambda x^\alpha + \mu y^\alpha)$. (12)

**Corollary 2.10:** $E_\alpha(\lambda x^\alpha) \otimes E_\alpha(\mu x^\alpha) = E_\alpha((\lambda + \mu)x^\alpha)$. (13)

**Remark 2.11:** Peng and Li [20] give an example to show that $E_\alpha(\lambda x^\alpha) \cdot E_\alpha(\lambda x^\alpha) = E_\alpha(\lambda(x+y)^\alpha)$ is not true for $0 < \alpha < 1$. On the other hand, Area, *et al*. [21] also provide a counterexample for $E_\alpha(\lambda x^\alpha) \cdot E_\alpha(\mu x^\alpha) = E_\alpha((\lambda + \mu)x^\alpha)$, $0 < \alpha < 1$.

**Definition 2.12:** If $f(z) = \sum_{k=0}^{\infty} a_k z^k$, $g_\alpha(\mu x^\alpha) = \sum_{k=0}^{\infty} b_k p_k(\mu x^\alpha)$, then

$$f_{\otimes\alpha}\left(g_\alpha(\mu x^\alpha)\right) = \sum_{k=0}^{\infty} a_k \left(g_\alpha(\mu x^\alpha)\right)^{\otimes k}. \quad (14)$$

The following is the method we used in this article.

**Theorem 2.13 (product rule for fractional derivative)** ([19])**:** *Let* $0 < \alpha \le 1$, $\lambda, \mu$ *be complex numbers, and* $f_\alpha$ , $g_\alpha$ *be fractional function. Then*

$$\left({}_0D_x^\alpha\right)[f_\alpha(\lambda x^\alpha) \otimes g_\alpha(\mu x^\alpha)] = \left({}_0D_x^\alpha\right)[f_\alpha(\lambda x^\alpha)] \otimes g_\alpha(\mu x^\alpha) + f_\alpha(\lambda x^\alpha) \otimes \left({}_0D_x^\alpha\right)[g_\alpha(\mu x^\alpha)]. \quad (15)$$

## III. MAIN RESULT AND EXAMPLE

In the following, we obtain the general solution of the first order LFDE.

**Theorem 3.1:** *Let* $0 < \alpha \le 1$, $C$ *be a constant and* $P(x^\alpha), Q(x^\alpha)$ *be* $\alpha$*-integral functions. Then the first order linear* $\alpha$*-fractional differential equation*

$${}_aD_x^\alpha[y(x^\alpha)] + P(x^\alpha) \otimes y(x^\alpha) = Q(x^\alpha) \quad (16)$$

*has the general solution*

$$y(x^\alpha) = E_\alpha\left(-\,{}_aI_x^\alpha[P(x^\alpha)]\right) \otimes \left({}_aI_x^\alpha\left[Q(x^\alpha) \otimes E_\alpha\left({}_aI_x^\alpha[P(x^\alpha)]\right)\right] + C\right). \quad (17)$$

***Proof*** *Step 1*. If $Q(x^\alpha) = 0$, then Eq. (16) becomes the homogeneous first order LFDE

$${}_aD_x^\alpha[y(x^\alpha)] + P(x^\alpha) \otimes y(x^\alpha) = 0. \quad (18)$$

Using integrating factor method, both sides of Eq. (18) are multiplied by $E_\alpha\left(-\,{}_aI_x^\alpha[P(x^\alpha)]\right)$ yields

$$E_\alpha\left({}_aI_x^\alpha[P(x^\alpha)]\right) \otimes \left({}_aD_x^\alpha[y(x^\alpha)] + P(x^\alpha) \otimes y(x^\alpha)\right) = 0. \quad (19)$$

By product rule for fractional derivatives, we get

$${}_aD_x^\alpha\left[E_\alpha\left({}_aI_x^\alpha[P(x^\alpha)]\right) \otimes y(x^\alpha)\right] = 0. \quad (20)$$

Therefore, we obtain the general solution of Eq. (19)

$$y(x^\alpha) = C \cdot E_\alpha\left(-\,{}_aI_x^\alpha[P(x^\alpha)]\right), \quad (21)$$

where $C$ is a constant.

*Step 2*. If $Q(x^\alpha) \neq 0$, then Eq. (16) is the non-homogeneous first order LFDE. By constant variation method, let

$$y(x^\alpha) = u(x^\alpha) \otimes E_\alpha\left(-\,{}_aI_x^\alpha[P(x^\alpha)]\right). \quad (22)$$

Using product rule for fractional derivative yields

$${}_aD_x^\alpha[y(x^\alpha)] = {}_aD_x^\alpha[u(x^\alpha)] \otimes E_\alpha\left(-\,{}_aI_x^\alpha[P(x^\alpha)]\right) - u(x^\alpha) \otimes P(x^\alpha) \otimes E_\alpha\left(-\,{}_aI_x^\alpha[P(x^\alpha)]\right). \quad (23)$$

Replace Eqs. (22), (23) into Eq. (16), we have

$${}_aD_x^\alpha[u(x^\alpha)] \otimes E_\alpha\left(-\,{}_aI_x^\alpha[P(x^\alpha)]\right) - u(x^\alpha) \otimes P(x^\alpha) \otimes E_\alpha\left(-\,{}_aI_x^\alpha[P(x^\alpha)]\right)$$
$$+P(x^\alpha) \otimes u(x^\alpha) \otimes E_\alpha\left(-\,{}_aI_x^\alpha[P(x^\alpha)]\right) = Q(x^\alpha). \quad (24)$$

It follows that

$$ {}_aD_x^{\alpha}[u(x^{\alpha})] \otimes E_{\alpha}\left(-\,{}_aI_x^{\alpha}[P(x^{\alpha})]\right) = Q(x^{\alpha}). \quad (25)$$

Thus,

$$ {}_aD_x^{\alpha}[u(x^{\alpha})] = Q(x^{\alpha}) \otimes E_{\alpha}\left({}_aI_x^{\alpha}[P(x^{\alpha})]\right). \quad (26)$$

Integrating both sides of Eq. (26) yields

$$ u(x^{\alpha}) = {}_aI_x^{\alpha}\left[Q(x^{\alpha}) \otimes E_{\alpha}\left({}_aI_x^{\alpha}[P(x^{\alpha})]\right)\right] + C. \quad (27)$$

Replace Eq. (27) into Eq. (22), then Eq. (17) is obtained. Q.e.d.

Next, we give an example to illustrate our result.

**Example 3.2:** Consider the first order linear $1/2$-fractional differential equation

$$ {}_0D_x^{1/2}\left[y\left(x^{1/2}\right)\right] + cos_{1/2}\left(x^{1/2}\right) \otimes y\left(x^{1/2}\right) = E_{1/2}\left(-sin_{1/2}\left(x^{1/2}\right)\right). \quad (28)$$

Its general solution is

$$ y\left(x^{1/2}\right) = E_{1/2}\left(-\,{}_0I_x^{1/2}\left[cos_{1/2}\left(x^{1/2}\right)\right]\right) \otimes \left({}_0I_x^{1/2}\left[E_{1/2}\left(-sin_{1/2}\left(x^{1/2}\right)\right) \otimes E_{1/2}\left({}_0I_x^{1/2}\left[cos_{1/2}\left(x^{1/2}\right)\right]\right)\right] + C\right)$$
$$ = \left(\frac{1}{\Gamma(3/2)}x^{1/2} + C\right) \otimes E_{1/2}\left(-sin_{1/2}\left(x^{1/2}\right)\right). \quad (29)$$

## IV. CONCLUSION

The general solution of first order LFDE can be obtained by using product rule, integrating factor and constant variation method. In fact, it is the generalization of first order linear ordinary differential equation. On the other hand, the new multiplication we defined is a natural operation in fractional calculus. In the future, we will use the Jumarie type of modified R-L fractional derivatives and the new multiplication to extend the research topics to the problems of engineering and applied science.

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