ISSN 2348-1196 (print)
International Journal of Computer Science and Information Technology Research ISSN 2348-120X (online)
Vol. 8, Issue 4, pp: (46-49), Month: October - December 2020, Available at: www.researchpublish.com

Fractional Clairaut's Differential Equation and Its Application

Chii-Huei Yu

School of Mathematics and Statistics, Zhaoqing University, Guangdong Province, China

Abstract: **This paper uses a new multiplication of fractional functions and the product rule and chain rule for fractional derivatives, regarding the Jumarie type of modified Riemann-Liouville (R-L) fractional derivative, to obtain the general solution and singular solution of fractional Clairaut's differential equation. On the other hand, an example is proposed to illustrate our results.**

Keyword: **New multiplication, Product rule, Chain rule, Modified R-L fractional derivative, Fractional Clairaut's differential equation.**

I. INTRODUCTION

Fractional calculus belongs to the field of mathematical analysis which involves the investigation and applications of integrals and derivatives of arbitrary order. Although fractional calculus has almost the same long history as the classical calculus, it was only in recent decades that its theory and applications have rapidly developed. Oldham and Spanier [1] published the first monograph in 1974. Ross [2] edited the first proceedings that was published in 1975. Thereafter theory and applications of fractional calculus have attracted much interest and have become a vibrant research area. Nowadays, the number of monographs and proceedings devoted to fractional calculus is already large, e.g. [3-8].

Fractional differential equations arise in many complex systems in nature and society with many dynamics, such as charge transport in amorphous semiconductors, the spread of contaminants in underground water, relaxation in viscoelastic materials like polymers, the diffusion of pollution in the atmosphere, and many more [9-10]. However, the problem of studying fractional differential equations has been dealt with by numerous authors throughout history, particularly in recent years [11-12]. A wide description of the existence and uniqueness of solutions of initial value problem for fractional order differential equations together with its applications can be found in the literature [13]. Other papers on fractional differential equations can refer to [15-21]. Unlike standard calculus, there is no unique definition of derivation and integration in fractional calculus. The commonly used definition is the Riemann-Liouville (R-L) fractional derivative. Other useful definitions include Caputo definition of fractional derivative, the Grunwald-Letinikov (G-L) fractional derivative, and Jumarie's modified R-L fractional derivative is used to avoid nonzero fractional derivative of constant functions.

In this paper, the general solution and singular solution of fractional Clairaut's differential equation can be obtained by using a new multiplication of fractional functions, and product rule and chain rule for fractional derivatives, regarding the Jumarie type of modified R-L fractional derivative. Moreover, the singular solution of fractional Clairaut's differential equation is the fractional envelope of the general solution curve family. In fact, the result we obtained is the generalization of Clairaut's ordinary differential equation. On the other hand, we provide an example to demonstrate the application of our results.

II. PRELIMINARIES AND RESULTS

Firstly, we introduce the fractional calculus used in this article.

Definition 2.1: Let α be a real number and m be a positive integer. Then the modified R-L fractional derivatives of Jumarie type is defined by ([14])

$$ {}_aD_x^{\alpha}[f(x)] = \begin{cases} \frac{1}{\Gamma(-\alpha)}\int_a^x (x-\tau)^{-\alpha-1} f(\tau)d\tau, & \text{if } \alpha < 0 \\ \frac{1}{\Gamma(1-\alpha)}\frac{d}{dx}\int_a^x (x-\tau)^{-\alpha}[f(\tau)-f(a)]\,d\tau & \text{if } 0 \le \alpha < 1 \\ \frac{d^m}{dx^m}\left({}_aD_x^{\alpha-m}\right)[f(x)], & \text{if } m \le \alpha < m+1 \end{cases} \quad (1) $$

where $\Gamma(y) = \int_0^\infty t^{y-1}e^{-t}dt$ is the gamma function defined on $y > 0$. If $\left({}_aD_x^{\alpha}\right)^n[f(x)] = \left({}_aD_x^{\alpha}\right)\left({}_aD_x^{\alpha}\right)\cdots\left({}_aD_x^{\alpha}\right)[f(x)]$ exists, then $f(x)$ is called n-th order α-fractional differentiable function, and $\left({}_aD_x^{\alpha}\right)^n[f(x)]$ is the n-th order α-fractional derivative of $f(x)$.

Proposition 2.2: *Suppose that α, β, c are real constants and $0 < \alpha \le 1$, then*

$$ {}_0D_x^{\alpha}\left[x^{\beta}\right] = \frac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha+1)}x^{\beta-\alpha}, \text{ if } \beta \ge \alpha \quad (2) $$

$$ {}_0D_x^{\alpha}[c] = 0, \quad (3) $$

In the following, we define a new multiplication of fractional functions.

Definition 2.3 ([15])**:** Suppose that λ, μ, z are complex numbers, $0 < \alpha \le 1$, j, l, k are non-negative integers, and a_k, b_k are real numbers, $p_k(z) = \frac{1}{\Gamma(k\alpha+1)}z^k$ for all k. The $\otimes$ multiplication is defined by

$$ p_j(\lambda x^{\alpha}) \otimes p_l(\mu y^{\alpha}) = \frac{1}{\Gamma(j\alpha+1)}(\lambda x^{\alpha})^j \otimes \frac{1}{\Gamma(l\alpha+1)}(\mu y^{\alpha})^l = \frac{1}{\Gamma((j+l)\alpha+1)}\binom{j+l}{j}(\lambda x^{\alpha})^j(\mu y^{\alpha})^l, \quad (4) $$

where $\binom{j+l}{j} = \frac{(j+l)!}{j!l!}$.

If $f_\alpha(\lambda x^{\alpha})$ and $g_\alpha(\mu y^{\alpha})$ are two fractional functions,

$$ f_\alpha(\lambda x^{\alpha}) = \sum_{k=0}^{\infty} a_k\, p_k(\lambda x^{\alpha}) = \sum_{k=0}^{\infty}\frac{a_k}{\Gamma(k\alpha+1)}(\lambda x^{\alpha})^k, \quad (5) $$

$$ g_\alpha(\mu y^{\alpha}) = \sum_{k=0}^{\infty} b_k\, p_k(\mu y^{\alpha}) = \sum_{k=0}^{\infty}\frac{b_k}{\Gamma(k\alpha+1)}(\mu y^{\alpha})^k, \quad (6) $$

then we define

$$ \begin{aligned} f_\alpha(\lambda x^{\alpha}) \otimes g_\alpha(\mu y^{\alpha}) &= \sum_{k=0}^{\infty} a_k\, p_k(\lambda x^{\alpha}) \otimes \sum_{k=0}^{\infty} b_k\, p_k(\mu y^{\alpha}) \\ &= \sum_{k=0}^{\infty}\left(\sum_{m=0}^{k} a_{k-m}b_m p_{k-m}(\lambda x^{\alpha}) \otimes p_m(\mu y^{\alpha})\right). \end{aligned} \quad (7) $$

Proposition 2.4: $f_\alpha(\lambda x^{\alpha}) \otimes g_\alpha(\mu y^{\alpha}) = \sum_{k=0}^{\infty}\frac{1}{\Gamma(k\alpha+1)}\sum_{m=0}^{k}\binom{k}{m} a_{k-m}b_m\,(\lambda x^{\alpha})^{k-m}(\mu y^{\alpha})^m.$ (8)

Definition 2.5: Let $\left(f_\alpha(\lambda x^{\alpha})\right)^{\otimes n} = f_\alpha(\lambda x^{\alpha}) \otimes \cdots \otimes f_\alpha(\lambda x^{\alpha})$ be the n times product of the fractional function $f_\alpha(\lambda x^{\alpha})$. If $f_\alpha(\lambda x^{\alpha}) \otimes g_\alpha(\lambda x^{\alpha}) = 1$, then $g_\alpha(\lambda x^{\alpha})$ is called the $\otimes$ reciprocal of $f_\alpha(\lambda x^{\alpha})$, and is denoted by $\left(f_\alpha(\lambda x^{\alpha})\right)^{\otimes -1}$.

Definition 2.6: If $f(z) = \sum_{k=0}^{\infty} a_k z^k$, $g_\alpha(\mu x^{\alpha}) = \sum_{k=0}^{\infty} b_k\, p_k(\mu x^{\alpha})$, then

$$ f_{\otimes\alpha}\left(g_\alpha(\mu x^{\alpha})\right) = \sum_{k=0}^{\infty} a_k \left(g_\alpha(\mu x^{\alpha})\right)^{\otimes k}. \quad (9) $$

Theorem 2.7 (product rule for fractional derivatives) ([15])**:** *Let* $0 < \alpha \le 1$, λ, μ *be complex numbers, and* f_α, g_α *be fractional function. Then*

$$ \left({}_0D_x^{\alpha}\right)[f_\alpha(\lambda x^{\alpha}) \otimes g_\alpha(\mu x^{\alpha})] = \left({}_0D_x^{\alpha}\right)[f_\alpha(\lambda x^{\alpha})] \otimes g_\alpha(\mu x^{\alpha}) + f_\alpha(\lambda x^{\alpha}) \otimes \left({}_0D_x^{\alpha}\right)[g_\alpha(\mu x^{\alpha})]. \quad (10) $$

Theorem 2.8 (chain rule for fractional derivatives) ([15])**:** *If* $f(z) = \sum_{k=0}^{\infty} a_k z^k$, $g_\alpha(\mu x^{\alpha}) = \sum_{k=0}^{\infty} b_k\, p_k(\mu x^{\alpha})$. *Let* $f_{\otimes\alpha}\left(g_\alpha(\mu x^{\alpha})\right) = \sum_{k=0}^{\infty} a_k \left(g_\alpha(\mu x^{\alpha})\right)^{\otimes k}$ *and* $f'_{\otimes\alpha}\left(g_\alpha(\mu x^{\alpha})\right) = \sum_{k=1}^{\infty} a_k\, k\left(g_\alpha(\mu x^{\alpha})\right)^{\otimes(k-1)}$, *then*

$$ \left({}_0D_x^{\alpha}\right)\left[f_{\otimes\alpha}\left(g_\alpha(\mu x^{\alpha})\right)\right] = f'_{\otimes\alpha}\left(g_\alpha(\mu x^{\alpha})\right) \otimes \left({}_0D_x^{\alpha}\right)[g_\alpha(\mu x^{\alpha})]. \quad (11) $$

Definition 2.9: Let $0 < \alpha \le 1$, the α-fractional Clairaut's differential equation is a first-order fractional differential equation

$$y^{\alpha} = \frac{1}{\Gamma(\alpha+1)} x^{\alpha} \otimes {}_0D_x^{\alpha}[y^{\alpha}] + f_{\alpha}\left({}_0D_x^{\alpha}[y^{\alpha}]\right), \tag{12}$$

where $f_{\alpha}(z_{\alpha})$ is a α-fractional function with $\left({}_0D_{z_{\alpha}}^{\alpha}\right)^2 [f_{\alpha}(z_{\alpha})] \neq 0$.

The following is the method for solving α-fractional Clairaut's differential equation:

Let $p_{\alpha} = {}_0D_x^{\alpha}[y^{\alpha}]$, then differentiating on both sides of Eq. (12) and by using product rule and chain rule for fractional derivatives yields

$$p_{\alpha} = p_{\alpha} + \frac{1}{\Gamma(\alpha+1)} x^{\alpha} \otimes {}_0D_x^{\alpha}[p_{\alpha}] + f'_{\otimes\alpha}(p_{\alpha}) \otimes {}_0D_x^{\alpha}[p_{\alpha}]. \tag{13}$$

Thus,

$$\left(\frac{1}{\Gamma(\alpha+1)} x^{\alpha} + f'_{\otimes\alpha}(p_{\alpha})\right) \otimes {}_0D_x^{\alpha}[p_{\alpha}] = 0. \tag{14}$$

If ${}_0D_x^{\alpha}[p_{\alpha}] = 0$, then $p_{\alpha} = c_{\alpha}$, and hence the general solution of Eq. (12) is

$$y^{\alpha} = \frac{c_{\alpha}}{\Gamma(\alpha+1)} x^{\alpha} + f_{\alpha}(c_{\alpha}), \tag{15}$$

where c_{α} is a constant.

If $\frac{1}{\Gamma(\alpha+1)} x^{\alpha} + f'_{\otimes\alpha}(p_{\alpha}) = 0$, then we get a singular solution of Eq. (12)

$$\begin{cases} x^{\alpha} = -\Gamma(\alpha+1) \cdot f'_{\otimes\alpha}(p_{\alpha}) \\ y^{\alpha} = -p_{\alpha} \cdot f'_{\otimes\alpha}(p_{\alpha}) + f_{\alpha}(p_{\alpha}) \end{cases}, \tag{16}$$

which are parametric equations of p_{α}.

III. APPLICATION

For the fractional Clairaut's differential equation discussed in this paper, an example is provided and we obtain its general solution and singular solution.

Example 3.1 Consider the $1/3$- fractional Clairaut's differential equation

$$y^{1/3} = \frac{1}{\Gamma\left(\frac{4}{3}\right)} x^{1/3} \otimes {}_0D_x^{1/3}\left[y^{1/3}\right] - \left({}_0D_x^{1/3}\left[y^{1/3}\right]\right)^{\otimes 2}. \tag{17}$$

That is, $f_{1/3}\left(p_{1/3}\right) = -p_{1/3}^2$ in Eq. (12). By Eq. (15), we have the general solution of Eq. (17),

$$y^{1/3} = \frac{c_{1/3}}{\Gamma\left(\frac{4}{3}\right)} x^{1/3} - c_{1/3}^2, \tag{18}$$

where $c_{1/3}$ is a constant.

And by Eq. (16), the singular solution of Eq. (17) is

$$\begin{cases} x^{1/3} = 2\Gamma\left(\frac{4}{3}\right) \cdot p_{1/3} \\ y^{1/3} = p_{1/3}^2 \end{cases}, \tag{19}$$

Thus, the singular solution is a fractional parabola $y^{1/3} = \frac{1}{4\left(\Gamma\left(\frac{4}{3}\right)\right)^2} x^{2/3}$, which is the the fractional envelope of the general solution curve family of Eq. (17).

IV. CONCLUSION

From the discussion above, we know that the product rule and chain rule for fractional derivatives is the major tools to find the general solution and singular solution of fractional Clairaut's differential equation. In fact, the applications of these two methods are extensive, and can be used to easily solve many difficult fractional differential equations; we endeavor to conduct further studies on related applications. In the future, we will extend the research topics to other fractional calculus and applied science problems. These results will be useful on the study of fractional differential equations and engineering mathematics.

REFERENCES

[1] K.B. Oldham and J. Spanier, The Fractional Calculus, Academic, New York, 1974.

[2] B. Ross (Ed.), Fractional Calculus and Its Applications, Springer-Verlag, Berlin, 1975.

[3] K.S. Miller and B. Ross, An Introduction to the Fractional Calculus and Fractional Differential Equations, Wiley, New York, 1993.

[4] A. Carpinteri and F. Mainardi (Eds.), Fractals and Fractional Calculus in Continuum Mechanics, Springer-Verlag, Wien/New York, 1997.

[5] C.A. Monje, Y.Q. Chen, B.M. Vinagre, D. Xue, and V. Feliu, Fractional-order Systems and Controls, Fundamentals and Applications, Springer, London, 2010

[6] F. Mainardi, Fractional Calculus and Waves in Linear Viscoelasticity, Imperial College, London, 2010.

[7] K. Diethelm, The Analysis of Fractional Differential Equations, Springer-Verlag, Berlin, 2010.

[8] J. Klafter, S.C. Lim, and R. Metzler (Eds.), Fractional Dynamics: Recent Advances, World Scientific, Singapore, 2011.

[9] I. Podlubny, Fractional Differential Equations, Academic, San Diego, 1999.

[10] A.A. Kilbas, H.M. Srivastava, and J.J. Trujillo, Theory and Applications of Fractional Differential Equations, Elsevier, Amsterdam, 2006.

[11] S., Guo, L., Mei, Y., Ling, Y., Sun,“The improved fractional sub-equation method and its applications to the space–time fractional differential equations in fluid mechanics, ” Physics Letters A Vol. 376, pp. 407–411, 2012.

[12] S. D., Eidelman, A. N., Kochubei,“Cauchy problem for fractional diffusion equations, ” Journal of Differential Equations, Vol. 199, pp. 211–255, 2004.

[13] S. G., Samko, A. A., Kilbas, O. I., Marichev, Fractional integrals and derivatives: theory and applications, Gordon and Breach science publishers, 1993.

[14] S. Das, Functional Fractional Calculus, 2nd Edition, Springer-Verlag, 2011.

[15] C. -H. Yu, “ Differential properties of fractional functions, ” International Journal of Novel Research in Interdisciplinary Studies, Vol. 7, No. 5, pp. 1-14, 2020.

[16] C. -H. Yu, “ Fractional derivatives of some fractional functions and their applications,” Asian Journal of Applied Science and Technology, Vol. 4, Issue 1, pp. 147-158, 2020.

[17] C. -H. Yu, “ Method for evaluating fractional derivatives of fractional functions,” International Journal of Scientific Research in Science, Engineering and Technology, Volume 7, Issue 4, pp. 286-290, 2020.

[18] C. -H. Yu, “ Integral form of particular solution of nonhomogeneous linear fractional differential equation with constant coefficients,” International Journal of Novel Research in Engineering and Science, Vol. 7, Issue 2, pp. 1-9, 2020.

[19] C. -H. Yu, “ Separable fractional differential equations,” International Journal of Mathematics and Physical Sciences Research, Vol. 8, Issue 2, pp. 30-34, 2020.

[20] C. -H. Yu, “A study of exact fractional differential equations,” International Journal of Interdisciplinary Research and Innovations, Vol. 8, Issue 4, pp. 100-105, 2020.

[21] C. -H. Yu, “ Research on first order linear fractional differential equations,” International Journal of Engineering Research and Reviews, Vol. 8, Issue 4, pp. 33-37, 2020.