International Journal of Mathematics and Physical Sciences Research ISSN 2348-5736 (Online)
Vol. 8, Issue 2, pp: (87-92), Month: October 2020 - March 2021, Available at: www.researchpublish.com

Adomian Decomposition Method to Solve the Second Order Ordinary Differential Equations

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Abstract: **Adomian decomposition method (ADM) is a type of semi-analysis method that is used to derive solutions for differential equations. PDE occurs in science and engineering, such as incompressible fluid flow, and in solving the Navier-Stokes and Poisson equation. It can be used to solve Cauchy problems of PDE that have initial condition problems. George Adomian from the University of Georgia developed this method. The objective of using this method to develop a unified theory to solve partial differential and second order differential equations. It can also be used to solve problems for stochastic systems by applying it to Ito integral (Adomian, 1986).**

Keywords: **Adomian Decomposition - Second Order Ordinary Differential Equations - Adomian polynomials.**

I. INTRODUCTION

An important aspect of ADM is that it facilitates the use of Adomian polynomials where solutions for non-linear part of the equations are converged without carrying out linearising of the system. ADM aim is for a unified theory to solve PDE, and this is followed the homotopy analysis method (Cherruault, 1989). An important feature is to develop and use Adomian polynomials that allow for convergence of solutions for the non-linear part of an equation. The polynomials provide the MacLaurin series for an external parameter that provides the solution with more flexibility than by using the Taylors series of expansion (García-Olivares, 2003). This paper presents a discussion on ADM and some examples where it is used.

II. ADOMIAN DECOMPOSITION METHOD

A brief discussion on ADM is presented below. Different forms and equations are given.

1st type:

Assume that that it is possible to use differential method for y = f(x) is sufficiently differentiable and the solution is (Adomian, 1986):

$$y' = f(x,y); \qquad y(a) = y_0 \qquad (1)$$

The above equation satisfied the Lipchitz condition.

If $L = \frac{d}{dx}$ with the inverse operator of L^{-1} is the one fold integral operations, it is written for the first order differential as:

$$L_x^{-1}(.) = \int_0^x (.)dx \qquad (2)$$

Let $L = \frac{d^2}{dx^2}$ with the inverse operator L^{-1} as a 2-fold integral operator, which is written as:

$$L_{xx}^{-1}(.) = \int_0^x \int_0^x (.)dxdx \qquad (3)$$

When the second order ordinary differential is considered than the numerical solution for the equation $y'' = f(x, y)$, it is written as:

$$y(x) = y_0 + y_1 x + L^{-1}[f(x, y)] \qquad (4)$$

In the above equation, $y_1 = y^1(x_0)$

$$\sum_{n=0}^{\infty} y_n(x) = y(x) \qquad (5)$$

Equation (5) is derived from ADM, and a series solution for *y (x)* is considered as an infinite addition of elements of (2)

$$y(x) = \sum_{n=0}^{\infty} y_n(x) \qquad (6)$$

Elements of $y(x)$ are calculated recursively. The ADM is used to define the *f (x, y)* with an infinite polynomial series.

$$f(x, y) = \sum_{n=0}^{\infty} A_n \qquad (7)$$

In (7), A_n is considered as Adomian polynomials, and it is calculated for different classes of non-linearity by:

$$A_n = \frac{1}{n!}\frac{d^n}{d\lambda N^n}\left[N\left(\sum_{k=0}^{n} N^k u_k\right)\right]_{N=0} \quad , n \geq 0 \qquad (8)$$

Simplifies to:

$$A_n = \sum_{k=1}^{n} C(k, n) f^{(k)}(U_0) \qquad n = 1,2,3, \ldots \qquad (9)$$

$$f^{(k)}(U_0) = \frac{d^k f(U_0)}{dU_0^k} \qquad (10)$$

In the above equation, *C (k, n)* are the sum of k components of *U*. The subscripts are added to n and divided by the repetition number of factorials.

When (6) ans (7) are placed in (4), the following equation is derived.

$$\sum_{n=0}^{\infty} y_n(x) = y_0 + y_1(x) + L^{-1}[\sum_{n=0}^{\infty} A_n\ ] \qquad (11)$$

Every term of the values for (11) is obtained from the equation:

$$y_0(x) = y_0 + y_1(x) \qquad (12)$$

and

$$y_{n+1}(x) = L^{-1} A_n\,, \qquad n \geq 0 \qquad (13)$$

2nd Type:

This type solves equations of the second order ordinary differential with constant coefficient of the type (Fadugba et al., 2013):

$$y'' + ay' + by = g(x) + f(x, y) \qquad (14)$$

$$y(0) = A, \quad y'(0) = B$$

In equation (14) *f (x, y)*, it is considered as a non-linear function

g (x) is the function to be solved

A, B, a, b are constants.

A new differential operator will be developed to study the problem given in (14).

For the transformation of *a - 2n + k* and *b = n (n+k)*, the transformation of equation (14) is given below with n, k as constants:

$$y" + (2n + k)\, y' + n\,(n + k)\, y = g\,(x) + f\,(x, y) \qquad (15)$$

The new differential operator is:

$$L(.) = e^{-nx}\frac{d}{dx}e^{-k}\frac{d}{dx}e^{n+k}(.) \qquad (16)$$

The equation (15) is now given as:

$$L\,(y) = g(x) + f\,(x, y) \qquad (17)$$

L^{-1} is the inverse operator and it is a 2-fold integral operator and it is given as:

$$L^{-1}\,(.) = e^{-(n+k)x}\int_0^x e^{kx}\int_0^x e^{nx}\,(.)dxdx \qquad (18)$$

When L^{-1} given in (18) are applied to the third terms of (15), the following equations are derived:

$$L^{-1}(y" + (2n + k)\, y' + n\,(n + k)\, y) = e^{-(n+k)x}\int_0^x e^{kx}\int_0^x e^{nx}\,(y" + (2n + k)\, y' + n\,(n + k)\, y)dxdx$$

$$= e^{-(n+k)x}\int_0^x e^{kx}\,(e^{nx}y' + (n + k)e^{nx}\, y - y'(0) - (n + k)y(0))dx$$

When L^{-1} is applied to (17), then we get:

$$y(x) = \frac{1}{k}y'(0)e^{-nx} + \frac{(n+k)}{k}y(0)e^{-nx} - \frac{1}{k}y'(0)e^{-(n+k)x} - \frac{(n)}{k}y(0)e^{-(n+k)x} + L^{-1}g(x) + L^{-1}f(x, y) \qquad (19)$$

With ADM, the solution *y(x)* with non-linear function *f(x,y)* for the infinite series is:

$$y(x) = \sum_{n=0}^{\infty} y_n(x) \qquad (20)$$

and also

$$f(x, y) = \sum_{n=0}^{\infty} A_n \qquad (21)$$

In the above equations, $y_n(x)$ for the solution *y(x)* are calculated on a recurrent basis. An algorithm is developed for this task and is given as:

$$A_0 = F(u)$$

$$A_1 = F'(u_0)u_1$$

$$A_2 = F'(u_0)u_2 + \frac{1}{2}F''(u_0)u_1^2 \qquad (22)$$

$$A_3 = F'(u_0)u_3 + F''(u_0)u_1u_2 + \frac{1}{3!}F''(u_0)u_1^3$$

III. APPLICATIONS

This section presents some numerical applications of the ADM methods detailed above.

1. Solving a homogenous linear problem

The method of solving such equations is given as follows (Hasan and Zhu, 2009)

$$y'' - 2y' = 0 \qquad (23)$$

$$y(0) = 0\,, y'(0) = 5$$

Input value for *n(n+k) = 2* and *2n+k = -2*

The $k = \pm 2i$

n = -1+-i where $i = \sqrt{-1}$

When the values of *n = -1 -i* and *k = 2i* are put in (16), then the operator obtained is:

$$L(.) = e^{(1+i)x}\frac{d}{dx}e^{-2ix}\frac{d}{dx}e^{(-1+i)x}(.)$$

or

$$L^{-1}\,(.) = \ e^{-(1-i)x}\int_0^x e^{2ix}\ \int_0^x e^{(-1-i)x}\,(.)dxdx$$

Equation (23) in operator form is:

$$L_y = 0 \qquad (24)$$

When L^{-1} is applied to (24), then:

$$L^{-1}L_y = 0$$

The exact solution is given below:

$$y = \frac{1}{2i}y'(0)e^{(1+i)x} + \frac{(-1+i)}{2i}y(0)e^{(1+i)x} + y(0)e^{(1+i)x} - \frac{1}{2i}y'(0)e^{(1-i)x} - \frac{(n1+i)}{2i}y(0)e^{(1-i)x}$$

$$= \frac{5i}{2}e^{x}(\cos x - i\sin x) - \frac{5i}{2}e^{x}(\cos x + i\sin x) = 5e^{x}\sin x$$

2. Solving the linear non-homogenous initial value problem

The linear non-homogenous initial value problem is to be derived for (Fadugba et al., 2013):

$$y'' - 3y' + 2y = x \qquad (25)$$

$$y(0) = 1\,, y'(0) = 0$$

Consider value of *n(n+k) = 2* and *2n+k = -3,*

then *k = -1* and *1*

n = -1 and *-2*

When *k=-1* and *n = -1* is input in (16), then the operator is turned into:

$$L(.) = e^{-2x}\frac{d}{dx}e^{4x}\frac{d}{dx}e^{-2x}(.)$$

Giving the equation

$$L^{-1}\,(.) = \ e^{2x}\int_0^x e^{-4x}\ \int_0^x e^{2x}\,(.)dxdx$$

This gives:

$$y(x) = -y'(0)e^{x} + 2y(0)e^{2x} + y(0)e^{2x} + y'(0)e^{2x} - 2y(0)e^{2x} + \frac{3}{4} - e^{x} + \frac{1}{4}e^{2x}$$

$$y(x) = \frac{3}{4} + e^{x} - \frac{3}{4}e^{2x} + \frac{x}{2}$$

3. Solving non-linear initial value problem

The initial value of the non-linear problem is (Hasan and Zhu, 2009):

$$y'' - 4y = 8y\, ln\, y \qquad (26)$$

$$y(0) = 1\,, y'(0) = 0$$

Input *n(n+k) = -4* and *2n + k = 0*

Then $n = \mp 2$ and $k = \pm 4$

When *n = 2* and *k = -4n* is input in (16), then:

$$L(.) = e^{2x}\frac{d}{dx}e^{4x}\frac{d}{dx}e^{-2x}(.)$$

As per (26), the equation is:

From (25), we have:

$$L\,y = 8y\, ln\, y$$

Using the earlier steps, the equation derived is:

$$y = \frac{-1}{4}y'(0)e^{-2x} + \frac{1}{2}y(0)e^{-2x} + y(0)e^{2x} + \frac{1}{4}y'(0)e^{2x} - \frac{1}{2}y(0)e^{2x}$$

$$= \frac{1}{2}e^{-2x} + \frac{1}{2}e^{2x}$$

$$y_{n+1} = L^{-1}A_n \quad , n \geq 0$$

In the above equation, A_n is the Adomian polynomial for the non-linear function $y\, ln\, y$

This is further reduced to:

$$A_0 = y_0 \ln y_0$$

$$A_1 = y_1\,(1 + \ln y_0)$$

$$A_2 = y_2\,(1 + \ln y_0) + (\frac{y_1^2}{2}(1 + \frac{1}{y_0})$$

Then

$$y_0 = 1 + 2x^2 + \frac{2}{3}x^4 + \cdots$$

$$y_1 = \frac{4}{3}x^4 + \frac{8}{9}x^6 + \cdots$$

$$y_2 = \frac{16}{45}x^6 + \frac{8}{15}x^8 + \cdots$$

So

$$y = y_0 + y_1 + y_2 + \cdots$$

$$= 1 + 2x^2 + 2x^4 + \frac{4}{3}x^6 + \frac{2}{3}x^8 + \cdots$$

$$y(x) = e^{2x^2}$$

IV. CONCLUSION

The paper examined the definition, methods, and four applications for ADM. This method was used to solve partial differential equations for a number of problems with initial conditions. The paper analysed and provided for four typical problems where ADM is used.

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