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THE HARD AND EASY WAY TO DETERMINE CLIQUE (A comparative Review)

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Abstract: **The field of Mathematics has managed to achieve unprecedented success across many fields. One of the most useful topics is clique, it has many potential applications especially in bioinformatics. Clique applications are becoming useful more and more nowadays. Many algorithms for finding cliques have been studied and being applied. In this article, we attempt to address the facts and wonders of reducing the procedure of finding cliques. We describe how to reduce the way of multiplying the rows and columns of matrices which is mostly boring to calculate.**

Keywords: **Clique, graph theory, directed graph, undirected graph, vertex, series, maximal subset, symmetric matrix.**

1. INTRODUCTION

It is a well-known fact that in the mathematical field of graph theory, Clique is one of the most useful and practical concepts, the problems of cliques are applicable in many fields ranging from computer science, mathematics, economics to social concepts.

The clique problem in the field of social science comes from the concept of considering a social network, which is the people know one another, where the graph's vertices represent people, and the graph's edges represent mutual acquaintance.

In computer science, the clique problem is the computational problem of finding a maximum clique, or all cliques, in a given graph.

Background

Graph theory: graph theory is a relatively new area of mathematics that is being widely used in formulating models in many problems in business, the social sciences, and the physical sciences. These applications include communications problems and the study of organizations and social structures.

Directed graph: a directed graph, or digraph, is a finite set of points $P_1, P_2, \ldots, P_n$, called vertices of nodes, together with a finite set of directed edges, each of which joins an ordered pair of distinct vertices.

Matrix: an mxn matrix A is a rectangular array of mn real (or complex) numbers arranged in m horizontal rows and n vertical columns:

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} \end{pmatrix}$$

- Turán's theorem gives a lower bound on the size of a clique in dense graphs.[3] If a graph has sufficiently many edges, it must contain a large clique. For instance, every graph with vertices and more than edges must contain a three-vertex clique.

- Ramsey's theorem states that every graph or its complement graph contains a clique with at least a logarithmic number of vertices.

- According to a result of Moon & Moser (1965), a graph with $3n$ vertices can have at most 3^n maximal cliques. The graphs meeting this bound are the Moon–Moser graphs $K_{3,3,\ldots}$, a special case of the Turán graphs arising as the extremal cases in Turán's theorem.

Topic Analysis

To all intents and purposes, The issues which we encounter to the multiplication of matrices relevant to the adjacency matrices and symmetric matrices are time and accuracy, for instance, if we have two matrices of 8x8 or 9x9 and so on, the multiplication of these matrices are much more difficult especially if need to find S^3 , hence the procedure is more complicated, we need to define the other way to calculate the S^3 easily, the way that I want to introduce is the multiplication of the first row of the first matrix with the first column of the second matrix, the multiplication of the second row of the first matrix with the second column of the second matrix and with the rest. Along these lines, we can obtain an effective way to determine clique and the result that will show the existence or inexistence of the clique in the given graph.

Definition:

In the mathematical area of graph theory, a **clique** is a subset of vertices of an undirected graph such that every two distinct vertices in the clique are adjacent; that is, its induced subgraph is complete.

But in directed graph clique is defined as below

A clique in a digraph is a subset S of the vertices satisfying the following properties:

a) S contains three or more vertices.

b) if P_i and P_j are in S, then there is a directed edge from P_i to P_j and a directed edge from P_j to P_i.

c) there is no larger subset T of the vertices that satisfy property (b) and contains S [that is, S is a maximal subset satisfying (b)].

Theorem

Let A(G) be the adjacency matrix of a digraph and $S = [S_{ij}]$ be the symmetric matrix, with $S^3 = [{S_{ij}}^3]$, where ${S_{ij}}^3$ is the ijth element in S^3. Then P_i belongs to a clique if and only if the diagonal entry ${S_{ij}}^3$ is positive.

The procedure for determining a clique in a digraph is as follows.

- Step 1. if A(G) is the adjacency matrix of the given digraph, compute the symmetric matrix $S = [S_{ij}]$, where

$$s_{ij} = s_{ji} = 1 \qquad if \qquad a_{ij} = a_{ji} = 1$$

Otherwise, $s_{ij} = 0$.

- Step 2. compute $S^3 = [{S_{ij}}^3]$.

- Step 3. P_i belongs to a clique if and only if ${S_{ij}}^3$ is positive.

For instance consider a digraph with five vertices $P_1, \quad P_2, \quad P_3, \quad P_4, \quad and \quad P_5$, adjacency matrix A(G) and symmetric matrix S, find the elements of the clique.

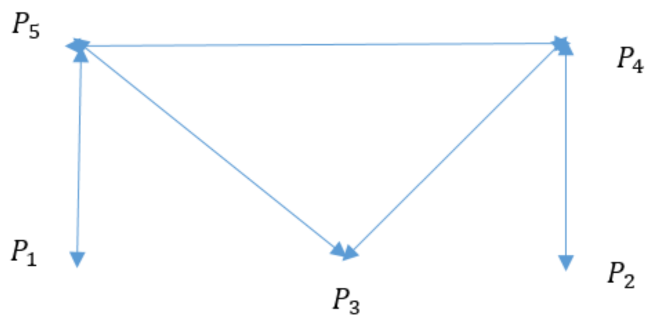

FIG. 1

Here, the set $\{P_1, P_2, P_3, P_4\ , P_5\}$ satisfies the condition a and c but it fails to satisfy condition b, because there is no directed edge between P_1 and P_i for $i \neq 5$ so, the set $\{P_1, P_2, P_3, P_4\ , P_5\}$ is not a clique.

The set S={ $P_3, P_4\ , P_5$} is a clique because it satisfies the conditions a, b and c successfully.

For large digraph, it is difficult to determine cliques using figures so we have to search for a way to determine cliques rather than the previous method. Fortunately, there is a way to determine cliques by using a matrix (symmetric matrix).

The following approach provides a useful method for detecting cliques that can easily be implemented on a computer. If $A(G) = [a_{ij}]$ is the given adjacency matrix of a digraph, form a new matrix $S = [s_{ij}]$:

$$s_{ij} = s_{ji} = 1 \qquad if \qquad a_{ij} = a_{ji} = 1$$

Otherwise, $s_{ij} = s_{ji} = 0$. Thus $s_{ij} = 1$ if P_i and P_j has access to each other. It should be noted that S is a symmetric matrix ($S = S^T$).

Theorem

Let $A(G)$ be the adjacency matrix of a digraph and $S = [s_{ij}]$ be the symmetric matrix defined above, with $S^3 = [{s_{ij}}^3]$, where ${s_{ij}}^3$ is the I, jth element in S^3. Then P_i belongs to a clique if and only if the diagonal entry ${s_{ij}}^3$ is positive.

Let us briefly consider why we examine the diagonal entries of S^3 in the theorem. First, note that the diagonal entry ${s_{ij}}^3$ of S^3 gives the number of way in which P_i has access to itself in three stages. If ${s_{ij}}^3 > 0$, then there is at least one way in which P_i has access to itself. Sine a digraph has no loops, this access must occur through two individuals: $P_i \rightarrow P_j \rightarrow P_k \rightarrow P_i$. Thus $s_{ij} \neq 0$. But $s_{ij} \neq 0$ implies that $s_{ji} \neq 0$, so $P_j \rightarrow P_i$. Similarly, any two of the individuals in $\{\ P_i, P_j\ , P_k\}$ have access to each other. This means that P_i, P_j and P_k all belong to the same clique. The opposite direction (if P_i is in a cliwue, then ${s_{ii}}^3 > 0$ is left.

For instance, we consider a digraph with adjacency with six vertices $P_1, \quad P_2, \quad P_3, \quad P_4, \quad P_5\ and \quad P_6$

matrix A(G) and symmetric matrix S, find the elements of the clique.

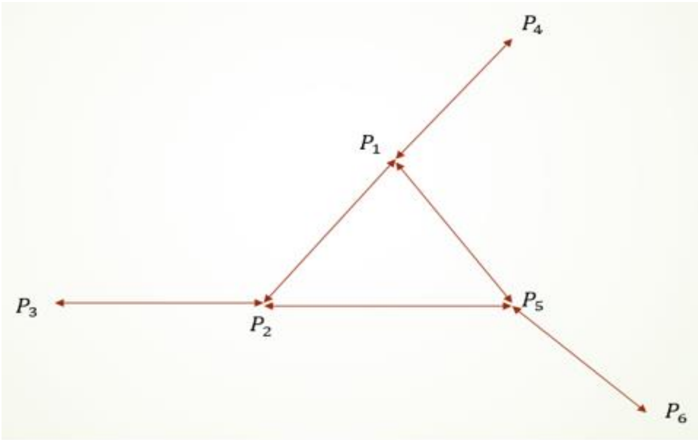

FIG. 2

The adjacency matrix A(G) and symmetric matrix S:

$$A(G) = \begin{array}{c} \\ P_1 \\ P_2 \\ P_3 \\ P_4 \\ P_5 \\ P_6 \end{array} \begin{array}{c} \begin{array}{cccccc} P_1 & P_2 & P_3 & P_4 & P_5 & P_6 \end{array} \\ \begin{bmatrix} 0 & 1 & 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix} \end{array} , \; S = \begin{array}{c} \\ P_1 \\ P_2 \\ P_3 \\ P_4 \\ P_5 \\ P_6 \end{array} \begin{array}{c} \begin{array}{cccccc} P_1 & P_2 & P_3 & P_4 & P_5 & P_6 \end{array} \\ \begin{bmatrix} 0 & 1 & 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix} \end{array}$$

$$SS = \begin{bmatrix} 0 & 1 & 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix} . \begin{bmatrix} 0 & 1 & 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 3 & 1 & 1 & 0 & 1 & 1 \\ 1 & 3 & 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 2 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$SS^2 = \begin{bmatrix} 0 & 1 & 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix} . \begin{bmatrix} 3 & 1 & 1 & 0 & 1 & 1 \\ 1 & 3 & 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 2 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 5 & 1 & 3 & 4 & 1 \\ 5 & 2 & 3 & 1 & 4 & 1 \\ 1 & 3 & 0 & 1 & 1 & 1 \\ 3 & 1 & 1 & 0 & 1 & 1 \\ 4 & 4 & 1 & 1 & 2 & 2 \\ 1 & 1 & 1 & 1 & 2 & 0 \end{bmatrix}$$

So, we can conclude that the elements of clique are P_1, P_2 and P_3. Therefore the set S as the set of elements of clique is S=$\{P_1, \; P_2, \; P_5\}$.

Now, the procedure of detecting the elements of a clique that we have seen in the above example was complicated enough, because first of all the multiplication of two matrices is a very long procedure and secondly the graphs may have a large number of vertices, for instance, a graph with 10, 12 or more vertices in this case the calculation will be more complicated, so here I have a suggestion to reduce the steps of determining the elements of clique, in the multiplication of S and S^2 we need to multiply the first row of the first matrix with the only first column of the second matrix in such manner the second row of the first matrix with the only second column of the second matrix and continue the same process until reaching to the last row of the first matrix with the only last column of the second matrix.

In other words in the result matrix (the multiplication of S and S^2) we need only the main diagonal of that matrix, and no need to calculate the other elements of the result matrix, similarly we will consider the same graph which is given above.

$$= \begin{matrix} & P_1 & P_2 & P_3 & P_4 & P_5 & P_6 \\ P_1 & 0 & 1 & 0 & 1 & 1 & 0 \\ P_2 & 1 & 0 & 1 & 0 & 1 & 0 \\ P_3 & 0 & 1 & 0 & 0 & 0 & 0 \\ P_4 & 1 & 0 & 0 & 0 & 0 & 0 \\ P_5 & 1 & 1 & 0 & 0 & 0 & 1 \\ P_6 & 0 & 0 & 0 & 0 & 1 & 0 \end{matrix}, \quad S = \begin{matrix} & P_1 & P_2 & P_3 & P_4 & P_5 & P_6 \\ P_1 & 0 & 1 & 0 & 1 & 1 & 0 \\ P_2 & 1 & 0 & 1 & 0 & 1 & 0 \\ P_3 & 0 & 1 & 0 & 0 & 0 & 0 \\ P_4 & 1 & 0 & 0 & 0 & 0 & 0 \\ P_5 & 1 & 1 & 0 & 0 & 0 & 1 \\ P_6 & 0 & 0 & 0 & 0 & 1 & 0 \end{matrix}$$

$$SS = \begin{bmatrix} 0 & 1 & 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix} . \begin{bmatrix} 0 & 1 & 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 3 & 1 & 1 & 0 & 1 & 1 \\ 1 & 3 & 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 2 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \end{bmatrix}$$

So now we can eliminate some product of the matrices of S and S^2:

$$SS^2 = \begin{bmatrix} 0 & 1 & 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix} . \begin{bmatrix} 3 & 1 & 1 & 0 & 1 & 1 \\ 1 & 3 & 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 2 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 2 & & & & & \\ & 2 & & & & \\ & & 0 & & & \\ & & & 0 & & \\ & & & & 2 & \\ & & & & & 0 \end{bmatrix}$$

As we have seen the result of the multiplication of S and S^2, we can conclude that the second method is handier than the first method. The technique is extraordinarily important in reducing the complicated procedure of multiplying matrices. In this way, we can easily get the desired result whether the clique is existing in the graph or not.

As indicated, the result of the elements of the clique are (P_1, P_2 and P_5) same in both.

To have a better understanding we can consider the below digraph

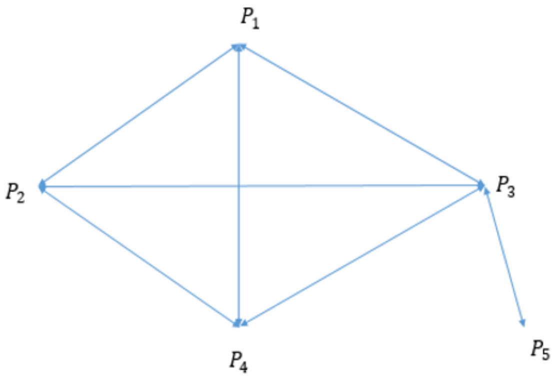

FIG. 3

And determine the elements of the clique if exists.

The adjacency matrix A(G) and symmetric matrix S:

$$A(G) = \begin{matrix} & P_1 & P_2 & P_3 & P_4 & P_5 \\ P_1 & 0 & 1 & 1 & 1 & 0 \\ P_2 & 1 & 0 & 1 & 1 & 0 \\ P_3 & 1 & 1 & 0 & 1 & 1 \\ P_4 & 1 & 1 & 1 & 0 & 0 \\ P_5 & 0 & 0 & 1 & 0 & 0 \end{matrix}, \quad S = \begin{matrix} & P_1 & P_2 & P_3 & P_4 & P_5 \\ P_1 & 0 & 1 & 1 & 1 & 0 \\ P_2 & 1 & 0 & 1 & 1 & 0 \\ P_3 & 1 & 1 & 0 & 1 & 1 \\ P_4 & 1 & 1 & 1 & 0 & 0 \\ P_5 & 0 & 0 & 1 & 0 & 0 \end{matrix}$$

$$S^2 = S \cdot S = \begin{bmatrix} 0 & 1 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix} . \begin{bmatrix} 0 & 1 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 3 & 2 & 2 & 2 & 1 \\ 2 & 3 & 2 & 2 & 1 \\ 2 & 2 & 4 & 2 & 0 \\ 2 & 2 & 2 & 3 & 1 \\ 1 & 1 & 0 & 1 & 1 \end{bmatrix}$$

Now

$$S^3 = S^2 \cdot S = \begin{bmatrix} 0 & 1 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix} . \begin{bmatrix} 3 & 2 & 2 & 2 & 1 \\ 2 & 3 & 2 & 2 & 1 \\ 2 & 2 & 4 & 2 & 0 \\ 2 & 2 & 2 & 3 & 1 \\ 1 & 1 & 0 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 5 & 7 & 8 & 7 & 2 \\ 7 & 6 & 8 & 7 & 2 \\ 8 & 8 & 6 & 8 & 4 \\ 7 & 7 & 8 & 6 & 2 \\ 2 & 2 & 4 & 2 & 0 \end{bmatrix}$$

The set $\{P_1, P_2, P_3\}$ satisfies conditions (a) and (b) for a clique, but it is not a clique, since it fails to satisfy condition (c). That is, $\{P_1, P_2, P_3\}$ contained in $\{P_1, P_2, P_3, P_4\}$, which satisfies conditions (a), (b) and (c). Thus the only clique in this digraph is $\{P_1, P_2, P_3, P_4\}$.

Now to avoid the complex calculations, we can do:

$$S^3 = S^2 \cdot S = \begin{bmatrix} 0 & 1 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix} . \begin{bmatrix} 3 & 2 & 2 & 2 & 1 \\ 2 & 3 & 2 & 2 & 1 \\ 2 & 2 & 4 & 2 & 0 \\ 2 & 2 & 2 & 3 & 1 \\ 1 & 1 & 0 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 5 & & & & \\ & 6 & & & \\ & & 6 & & \\ & & & 6 & \\ & & & & 0 \end{bmatrix}$$

It is obvious that the set of elements of the clique is $\{P_1, P_2, P_3, P_4\}$.

2. CONCLUSION

By surveying and comparing the multiplication of matrices to determine clique or specify the graph such that the graph is strongly connected or not, we can state the results as below:

1. Finding a way to determine a clique and reduce complex calculations and multiplication of matrices is not only serves to prevent time consumption but also serves high accuracy.

2. The way to determine that the graph is strongly connected or not is the same tough procedure.

3. If we multiply the two 9x9 matrices and apply for the given way we reduce the procedure 72 times, which is the most powerful way to accelerate the calculation.

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