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Study on Fractional Newton's Law of Cooling

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Abstract: **This paper makes use of Jumarie's modified Riemann-Liouville (R-L) fractional derivatives to study fractional Newton's law of cooling. The methods we used include a new multiplication, separation of variables, and chain rule for fractional derivatives. In addition, Mittag-Leffler function plays an important role in this paper, which is similar to the exponential function in classical calculus. On the other hand, an example is provided to illustrate our result.**

Keywords: **Jumarie's modified R-L fractional derivatives, fractional Newton's law of cooling, new multiplication, separation of variables, chain rule for fractional derivatives, Mittag-Leffler function, exponential function.**

I. INTRODUCTION

In a letter to L'Hospital in 1695, Leibniz proposed the possibility of generalizing classical differentiation to fractional order and asked what the result about $\frac{d^{1/2}x}{dx^{1/2}}$. After 124 years, Lacroix gave the right answer to this question for the first time that $\frac{d^{1/2}x}{dx^{1/2}} = \frac{2}{\sqrt{\pi}} x^{1/2}$. For a long time, due to the lack of practical application, fractional calculus has not been widely used. However, in the past few decades, fractional calculus has gained much attention as a result of its demonstrated applications in various fields of science and engineering such as physics, mathematical biology, mechanics, elasticity, dynamics, control theory, electronics, modelling, economics, chemistry [1-12]. But the definition of fractional derivative is not unique, there are many useful definitions include Riemann-Liouville (R-L) fractional derivative [13], Caputo fractional derivative [14], Grunwald-Letnikov fractional derivative [15], Jumarie's modification of R-L fractional derivative [16]. Riemann-Liouville definition the fractional derivative of a constant is non-zero which creates a difficulty to relate between the classical calculus. To overcome this difficulty, Jumarie [16] modified the definition of fractional derivative of Riemann-Liouville type and with this new formula, we obtain the derivative of a constant as zero. Thus, it is easier to connect fractional calculus with classical calculus by using this definition. Furthermore, by using the Jumarie modified definition of fractional derivative, we obtain the derivative of Mittag-Leffler function is Mittag-Leffler function itself. Like the classical derivative, the derivative of exponential function is exponential function itself. Therefore, the modified R-L fractional derivative of Jumarie type has a conjugate relationship with classical calculus. In many cases, it is easy to solve the fractional differential equations based on Jumarie fractional derivative [17-23]. On the other hand, all definitions of fractional derivatives satisfy the property of linearity. However, properties, such as the product rule, quotient rule, chain rule, Rolle's theorem, mean value theorem and composition rule and so on, they are lacking in almost all fractional derivatives. To avoid these difficulties, in [24], we provide a new multiplication to satisfy the above properties.

Newton's law of cooling suggests that the intensity of energy transfer in the form of heat depends on a difference of temperatures of the interacting physical systems. Linear character of this dependence is widely accepted. It is in good agreement with the reality in forced convection, and only holds in natural convection when the temperature difference is not too large. Newton's law of cooling was determined by Newton's experiment in 1701. To be precise, it states that the rate of heat loss of an object is proportional to the temperature difference between the surface of the object and the surrounding environment. Newton's law of cooling is one of the basic laws of heat transfer, used to calculate the amount of convective heat. It can be formulated as follows:

$$\frac{dT}{dt} = k(T(t) - T_m), \qquad (1)$$

where k is the Newton's cooling constant, T is the temperature of the surface of the object, and T_m is the temperature of the surrounding environment.

[27] considered Newton's law of cooling by using Riemann-Liouville and Caputo fractional derivatives, and has carried out experimental tests on different liquids. On the other hand, Ortega et al. [28] studied Newton's law of cooling with conformable derivatives. Almeida et al. [29-30] used some new fractional derivatives to obtain better results to fit the data of some modeling problems. In this paper, we consider Newton's law of cooling from a different perspective. Based on Jumarie type of modified R-L fractional derivatives, the fractional differential equation describing Newton's law of cooling is given by

$$\left({}_{t_0}D_t^{\alpha}\right)[T(t)] = k_{\alpha}(T(t) - T_{\alpha}), \qquad (2)$$

where ${}_{t_0}D_t^{\alpha}$ is the Jumarie's modified R-L fractional derivatives, k_{α} is the α-fractional Newton's cooling constant and T_{α} is the α-fractional temperature of the surrounding environment.

In this article, we use a new multiplication mentioned above, separation of variables, and chain rule for fractional derivatives to study fractional Newton's law of cooling. Moreover, the Mittag-Leffler function is very important in this paper.

II. PRELIMINARIES AND MAJOR RESULTS

In the following, the definition of fractional derivative used in this paper is introduced below.

Definition 2.1: If α is a real number and m is a positive integer. The Jumarie type modified R-L fractional derivatives ([16]) is defined by

$$ {}_{a}D_t^{\alpha}[f(t)] = \begin{cases} \frac{1}{\Gamma(-\alpha)}\int_a^x (t-\tau)^{-\alpha-1} f(\tau)d\tau, & \text{if } \alpha < 0 \\ \frac{1}{\Gamma(1-\alpha)}\frac{d}{dx}\int_a^x (t-\tau)^{-\alpha}[f(\tau) - f(a)]\,d\tau & \text{if } 0 \le \alpha < 1 \\ \frac{d^m}{dx^m}\left({}_{a}D_x^{\alpha-m}\right)[f(t)], & \text{if } m \le \alpha < m+1 \end{cases} \qquad (3)$$

where $\Gamma(w) = \int_0^{\infty} s^{w-1}e^{-s}ds$ is the gamma function defined on $w > 0$. Moreover, we define the fractional integral of (t), $\left({}_{a}I_t^{\alpha}\right)[f(t)] = \left({}_{a}D_t^{-\alpha}\right)[f(t)]$, where $\alpha > 0$ and $f(t)$ is called α-fractional integrable function.

Proposition 2.2 ([25])**:** *Suppose that α, β, c are real numbers and $\beta \ge \alpha > 0$, then*

$$ {}_{0}D_t^{\alpha}[t^{\beta}] = \frac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha+1)}t^{\beta-\alpha}, \qquad (4)$$

and

$$ {}_{0}D_x^{\alpha}[c] = 0. \qquad (5)$$

Definition 2.3 ([26])**:** The Mittag-Leffler function is defined by

$$E_{\alpha}(z) = \sum_{k=0}^{\infty}\frac{z^k}{\Gamma(k\alpha+1)}, \qquad (6)$$

where α is a real number, $\alpha > 0$, and z is a complex variable. $E_{\alpha}(t) = \sum_{k=0}^{\infty}\frac{t^{k\alpha}}{\Gamma(k\alpha+1)}$ is called the α-fractional exponential function.

In the following, we introduce a new multiplication of fractional functions.

Definition 2.4 ([24])**:** If λ, μ, z are complex numbers, $0 < \alpha \le 1$, j, l, n are non-negative integers, and a_n, b_n are real numbers, $p_n(z) = \frac{1}{\Gamma(n\alpha+1)}z^n$ for all n. The $\otimes$ multiplication is defined by

$$p_j(\lambda x^{\alpha}) \otimes p_l(\mu y^{\alpha}) = \frac{1}{\Gamma(j\alpha+1)}(\lambda x^{\alpha})^j \otimes \frac{1}{\Gamma(l\alpha+1)}(\mu y^{\alpha})^l = \frac{1}{\Gamma((j+l)\alpha+1)}\binom{j+l}{j}(\lambda x^{\alpha})^j(\mu y^{\alpha})^l, \qquad (7)$$

where $\binom{j+l}{j} = \frac{(j+l)!}{j!\,l!}$.

Let $f(\lambda x^\alpha)$ and $g(\mu y^\alpha)$ be two fractional functions,

$$f(\lambda x^\alpha)=\sum_{n=0}^{\infty}a_n\,p_n(\lambda x^\alpha)=\sum_{n=0}^{\infty}\frac{a_n}{\Gamma(n\alpha+1)}(\lambda x^\alpha)^n, \quad (8)$$

$$g(\mu y^\alpha)=\sum_{n=0}^{\infty}b_n\,p_n(\mu y^\alpha)=\sum_{n=0}^{\infty}\frac{b_n}{\Gamma(n\alpha+1)}(\mu y^\alpha)^n. \quad (9)$$

Then we define

$$f(\lambda x^\alpha)\otimes g(\mu y^\alpha)=\sum_{n=0}^{\infty}a_n\,p_n(\lambda x^\alpha)\otimes\sum_{n=0}^{\infty}b_n\,p_n(\mu y^\alpha)$$

$$=\sum_{n=0}^{\infty}\left(\sum_{m=0}^{n}a_{n-m}b_m p_{n-m}(\lambda x^\alpha)\otimes p_m(\mu y^\alpha)\right). \quad (10)$$

Proposition 2.5: $f(\lambda x^\alpha)\otimes g(\mu y^\alpha)=\sum_{n=0}^{\infty}\frac{1}{\Gamma(n\alpha+1)}\sum_{m=0}^{n}\binom{n}{m}a_{n-m}b_m(\lambda x^\alpha)^{n-m}(\mu y^\alpha)^m.$ (11)

Definition 2.6: Suppose that $\left(f(\lambda x^\alpha)\right)^{\otimes n}=f(\lambda x^\alpha)\otimes\cdots\otimes f(\lambda x^\alpha)$ is the n times product of the fractional function $f(\lambda x^\alpha)$. If $f(\lambda x^\alpha)\otimes g(\lambda x^\alpha)=1$, then $g(\lambda x^\alpha)$ is called the $\otimes$ reciprocal of $f(\lambda x^\alpha)$, and is denoted by $\left(f(\lambda x^\alpha)\right)^{\otimes -1}$.

Theorem 2.7 (chain rule for fractional derivatives) ([24])**:** *If* $f(z)=\sum_{n=0}^{\infty}a_n z^n$, $g_\alpha(\mu x^\alpha)=\sum_{n=0}^{\infty}b_n\,p_n(\mu x^\alpha)$. *Let* $f_{\otimes\alpha}\left(g_\alpha(\mu x^\alpha)\right)=\sum_{n=0}^{\infty}a_n\left(g_\alpha(\mu x^\alpha)\right)^{\otimes n}$ *and* $f'_{\otimes\alpha}\left(g_\alpha(\mu x^\alpha)\right)=\sum_{n=1}^{\infty}na_n\left(g_\alpha(\mu x^\alpha)\right)^{\otimes(n-1)}$, *then*

$$\left({}_0D_x^\alpha\right)\left[f_{\otimes\alpha}\left(g_\alpha(\mu x^\alpha)\right)\right]=f'_{\otimes\alpha}\left(g_\alpha(\mu x^\alpha)\right)\otimes\left({}_0D_x^\alpha\right)\left[g_\alpha(\mu x^\alpha)\right]. \quad (12)$$

The following is the main result in this article, we obtain a new exact solution for Newton's law of cooling with Jumarie's modified R-L fractional derivatives.

Theorem 2.8: *Let* $0<\alpha\le 1$, *then the fractional differential equation with an initial condition that describes Newton's law of cooling*

$$\begin{cases}\left({}_{t_0}D_t^\alpha\right)[T(t)]=k_\alpha(T(t)-T_\alpha)\\ \qquad T(t_0)=T_0\end{cases} \quad (13)$$

has the exact solution

$$T(t)=T_\alpha+(T_0-T_\alpha)\cdot E_\alpha\left(k_\alpha(t-t_0)\right), \quad (14)$$

where ${}_{t_0}D_t^\alpha$ *is the Jumarie's modified R-L fractional derivatives,* k_α *is the* α*-fractional Newton's cooling constant and* T_α *is the* α*-fractional temperature of the surrounding environment.*

Proof Rewrite $\left({}_{t_0}D_t^\alpha\right)[T(t)]=k_\alpha(T(t)-T_\alpha)$ into

$$\frac{d^\alpha T(t)}{[d(t-t_0)]^\alpha}=k_\alpha(T(t)-T_\alpha). \quad (15)$$

By separation of variables, we have

$$\left({}_{t_0}J_t^\alpha\right)\left[(T(t)-T_\alpha)^{\otimes -1}d^\alpha T(t)\right]=\left({}_{t_0}J_t^\alpha\right)\left[k_\alpha[d(t-t_0)]^\alpha\right]. \quad (16)$$

It follows that

$$Ln_\alpha(T(t)-T_\alpha)=k_\alpha\cdot\frac{1}{\Gamma(\alpha+1)}(t-t_0)^\alpha+C_1, \quad (17)$$

where C_1 is some constant.

And hence,

$$T(t)-T_\alpha=C\cdot E_\alpha\left(k_\alpha(t-t_0)\right) \quad (18)$$

for some constant C.

Since $T(t_0)=T_0$, we obtain $C=T_0-T_\alpha$. It follows that the desired result holds. Q.e.d.

Remark 2.9: In Eq. (16),

$$\left({}_{t_0}J_t^\alpha\right)\left[(T(t)-T_\alpha)^{\otimes -1}d^\alpha T(t)\right]=\left({}_{t_0}I_t^\alpha\right)\left[(T(t)-T_\alpha)^{\otimes -1}\right], \quad (19)$$

and

$$\left({}_{t_0}J_t^{\alpha}\right)[k_\alpha[d(t-t_0)]^{\alpha}] = \left({}_{t_0}I_t^{\alpha}\right)[k_\alpha]. \quad (20)$$

On the other hand, it is easy to verify that Eq. (14) satisfies Eqs. (13). Since by chain rule for fractional derivatives, it is easy to know that $T(t) = T_\alpha + (T_0 - T_\alpha)\cdot E_\alpha\big(k_\alpha(t-t_0)\big)$ satisfies

$$\left({}_{t_0}I_t^{\alpha}\right)\left[T_\alpha + (T_0 - T_\alpha)\cdot E_\alpha\big(k_\alpha(t-t_0)\big)\right]$$
$$= (T_0 - T_\alpha)k_\alpha \cdot E_\alpha\big(k_\alpha(t-t_0)\big)$$
$$= k_\alpha(T(t) - T_\alpha),$$

and $T(t_0) = T_0$.

III. EXAMPLES

Example 3.1: The thermometer is taken from 20 ℃ indoor to 5 ℃ outdoor. After one minute, the thermometer reads 12 ℃. (1) What's the thermometer reading in another minute? (2) How long does it take for the thermometer to read 6 ℃ ?

Solution Let $T(t)$ be the reading of the thermometer after taking t minutes from the indoor to the outside. By fractional Newton's law of cooling, we obtain the following fractional differential equation with an initial condition

$$\begin{cases}\left({}_0D_t^{\alpha}\right)[T(t)] = k_\alpha(T(t)-5) \\ T(0) = 20\end{cases}. \quad (21)$$

Using Theorem 2.8 yields

$$T(t) = 5 + 15\cdot E_\alpha(k_\alpha t). \quad (22)$$

From the condition that the thermometer reading is 12 ℃ after one minute, we can obtain

$$12 = 5 + 15\cdot E_\alpha(k_\alpha). \quad (23)$$

And hence,

$$k_\alpha = Ln_\alpha\left(\frac{7}{15}\right). \quad (24)$$

Therefore,

$$T(t) = 5 + 15\cdot E_\alpha\left(Ln_\alpha\left(\frac{7}{15}\right)t\right). \quad (25)$$

Substituting $t = 2$ into Eq. (25), we get

$$T(2) = 5 + 15\cdot E_\alpha\left(2Ln_\alpha\left(\frac{7}{15}\right)\right). \quad (26)$$

On the other hand, let $T(t) = 6$, i.e.,

$$5 + 15\cdot E_\alpha\left(Ln_\alpha\left(\frac{7}{15}\right)t\right) = 6. \quad (27)$$

Then

$$E_\alpha\left(Ln_\alpha\left(\frac{7}{15}\right)t\right) = \frac{1}{15}. \quad (28)$$

Thus,

$$t = \frac{Ln_\alpha\left(\frac{1}{15}\right)}{Ln_\alpha\left(\frac{7}{15}\right)}. \quad (29)$$

IV. CONCLUSION

From the above discussion, we know that the new multiplication we defined, separation of variables, and chain rule for fractional derivatives play important roles in this study. In fact, these methods are widely used and can easily solve many problems about fractional ordinary differential equations and fractional partial differential equations. In addition, the fractional Newton's law of cooling is a generalization of the classical Newton's law of cooling. In the future, we will take advantage of the Jumarie type of modified R-L fractional derivatives and the new multiplication we defined to extend our research fields to applied mathematics and engineering problems.

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