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Fractional Mean Value Theorem and Its Applications

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Abstract: **Based on the Jumarie type of modified Riemann-Liouville (R-L) fractional derivatives, the method used in this paper is to first transform the definition of modified R-L fractional derivatives into the form of limit, and then use fractional Fermat's theorem and fractional Rolle's theorem to prove our main result: fractional mean value theorem. In fact, this result is the generalization of mean value theorem for classical calculus. On the other hand, we provide some examples to illustrate the applications of fractional mean value theorem.**

Keywords: **Jumarie type of modified R-L fractional derivatives, form of limit, fractional Fermat's theorem, fractional Rolle's theorem, fractional mean value theorem.**

I. INTRODUCTION

Fractional calculus belongs to the field of mathematical analysis, involving the research and applications of arbitrary order integrals and derivatives. Fractional calculus originated from a problem put forward by L'Hospital and Leibniz in 1695. Therefore, the history of fractional calculus was formed more than 300 years ago, and fractional calculus and classical calculus have almost the same long history. Since then, fractional calculus has attracted the attention of many contemporary great mathematicians, such as N. H. Abel, M. Caputo, L. Euler, J. Fourier, A. K. Grunwald, J. Hadamard, G. H. Hardy, O. Heaviside, H. J. Holmgren, P. S. Laplace, G. W. Leibniz, A. V. Letnikov, J. Liouville, B. Riemann, M. Riesz, and H. Weyl. With the efforts of researchers, the theory of fractional calculus and its applications have developed rapidly. On the other hand, fractional calculus has wide applications in continuum mechanics, quantum mechanics, electrical engineering, fluid science, viscoelasticity, control theory, dynamics, finance, and so on [4-20, 29]. Moreover, the applications of fractional calculus to fractional differential equations can refer to [21-28].

However, different from the traditional calculus, the rule of fractional derivative is not unique, many scholars have given the definitions of fractional derivatives. The common definition is Riemann-Liouville (R-L) fractional derivatives [1-2]. Other useful definitions include Caputo fractional derivatives, Grunwald-Letnikov (G-L) fractional derivatives [1], and Jumarie type of R-L fractional derivatives to avoid non-zero fractional derivative of constant function [3]. In this paper, we make use of fractional Fermat's theorem and fractional Rolle's theorem to prove our major result: fractional mean value theorem. In fact, the fractional mean value theorem is the generalization of mean value theorem in traditional calculus. In addition, two examples are proposed to illustrate its applications.

II. METHODS AND RESULTS

The following is the fractional calculus used in this article.

Definition 2.1: Assume that α is a real number and m is a positive integer. The modified Riemann-Liouville fractional derivatives of Jumarie type ([12]) is defined by

$$
{}_{x_0}D_x^{\alpha}[f(x)] = \begin{cases} \frac{1}{\Gamma(-\alpha)}\int_{x_0}^{x}(x-\tau)^{-\alpha-1}f(\tau)d\tau, & \text{if } \alpha<0 \\ \frac{1}{\Gamma(1-\alpha)}\frac{d}{dx}\int_{x_0}^{x}(x-\tau)^{-\alpha}[f(\tau)-f(a)]\,d\tau & \text{if } 0\le\alpha<1 \\ \frac{d^m}{dx^m}\left({}_{x_0}D_x^{\alpha-m}\right)[f(x)], & \text{if } m\le\alpha<m+1 \end{cases} \quad (1)
$$

where $\Gamma(w)=\int_0^\infty t^{w-1}e^{-t}\,dt$ is the gamma function defined on $w>0$. If $\left({}_{x_0}D_x^\alpha\right)^n[f(x)]=\left({}_{x_0}D_x^\alpha\right)\left({}_{x_0}D_x^\alpha\right)\cdots\left({}_{x_0}D_x^\alpha\right)[f(x)]$ exists, then $f(x)$ is called n-th order α-fractional differentiable function, and $\left({}_{x_0}D_x^\alpha\right)^n[f(x)]$ is the n-th order α-fractional derivative of $f(x)$. We note that $\left({}_{x_0}D_x^\alpha\right)^n\neq {}_{x_0}D_x^{n\alpha}$ in general. We have the following property [13].

Proposition 2.2: *Let α,β,c be real numbers and $\beta\geq\alpha>0$, then*

$$ {}_0D_x^\alpha\left[x^\beta\right]=\frac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha+1)}x^{\beta-\alpha}, \tag{2}$$

and

$$ {}_0D_x^\alpha[c]=0. \tag{3}$$

Definition 2.3 ([14])**:** The Mittag-Leffler function is defined by

$$E_\alpha(z)=\sum_{k=0}^\infty\frac{z^k}{\Gamma(k\alpha+1)}, \tag{4}$$

where α is a real number, $\alpha>0$, and z is a complex variable.

Definition 2.4 ([13])**:** Suppose that $0<\alpha\leq 1$ and x is a real variable. Then $E_\alpha(x^\alpha)$ is called α-order fractional exponential function, and the α-order fractional cosine and sine function are defined as follows:

$$cos_\alpha(x^\alpha)=\sum_{k=0}^\infty\frac{(-1)^k x^{2k\alpha}}{\Gamma(2k\alpha+1)}, \tag{5}$$

and

$$sin_\alpha(x^\alpha)=\sum_{k=0}^\infty\frac{(-1)^k x^{(2k+1)\alpha}}{\Gamma((2k+1)\alpha+1)}. \tag{6}$$

In the following, we introduce a new multiplication of fractional functions.

Definition 2.5 ([15])**:** If λ,μ,z are complex numbers, $0<\alpha\leq 1$, j,l,k are non-negative integers, and a_k,b_k are real numbers, $p_k(z)=\frac{1}{\Gamma(k\alpha+1)}z^k$ for all k. The $\otimes$ multiplication is defined by

$$p_j(\lambda x^\alpha)\otimes p_l(\mu y^\alpha)=\frac{1}{\Gamma(j\alpha+1)}(\lambda x^\alpha)^j\otimes\frac{1}{\Gamma(l\alpha+1)}(\mu y^\alpha)^l=\frac{1}{\Gamma((j+l)\alpha+1)}\binom{j+l}{j}(\lambda x^\alpha)^j(\mu y^\alpha)^l, \tag{7}$$

where $\binom{j+l}{j}=\frac{(j+l)!}{j!\,l!}$.

If $f(\lambda x^\alpha)$ and $g(\mu y^\alpha)$ are two fractional functions,

$$f(\lambda x^\alpha)=\sum_{k=0}^\infty a_k\,p_k(\lambda x^\alpha)=\sum_{k=0}^\infty\frac{a_k}{\Gamma(k\alpha+1)}(\lambda x^\alpha)^k, \tag{8}$$

$$g(\mu y^\alpha)=\sum_{k=0}^\infty b_k\,p_k(\mu y^\alpha)=\sum_{k=0}^\infty\frac{b_k}{\Gamma(k\alpha+1)}(\mu y^\alpha)^k, \tag{9}$$

then we define

$$f(\lambda x^\alpha)\otimes g(\mu y^\alpha)=\sum_{k=0}^\infty a_k\,p_k(\lambda x^\alpha)\otimes\sum_{k=0}^\infty b_k\,p_k(\mu y^\alpha)$$
$$=\sum_{k=0}^\infty\left(\sum_{m=0}^k a_{k-m}b_m p_{k-m}(\lambda x^\alpha)\otimes p_m(\mu y^\alpha)\right). \tag{10}$$

Proposition 2.6: $f(\lambda x^\alpha)\otimes g(\mu y^\alpha)=\sum_{k=0}^\infty\frac{1}{\Gamma(k\alpha+1)}\sum_{m=0}^k\binom{k}{m}a_{k-m}b_m(\lambda x^\alpha)^{k-m}(\mu y^\alpha)^m.$ (11)

Definition 2.7: Let $\left(f(\lambda x^\alpha)\right)^{\otimes n}=f(\lambda x^\alpha)\otimes\cdots\otimes f(\lambda x^\alpha)$ be the n times product of the fractional function $f(\lambda x^\alpha)$. If $f(\lambda x^\alpha)\otimes g(\lambda x^\alpha)=1$, then $g(\lambda x^\alpha)$ is called the $\otimes$ reciprocal of $f(\lambda x^\alpha)$, and is denoted by $\left(f(\lambda x^\alpha)\right)^{\otimes -1}$.

Definition 2.8: If $f(z)=\sum_{k=0}^\infty a_k z^k$, $g(\mu x^\alpha)=\sum_{k=0}^\infty b_k\,p_k(\mu x^\alpha)$, then

$$f_\otimes\left(g(\mu x^\alpha)\right)=\sum_{k=0}^\infty a_k\left(g(\mu x^\alpha)\right)^{\otimes k}. \tag{12}$$

Next, we transform the definition of Jumarie type of modified R-L fractional derivatives into the form of limit.

Theorem 2.9: *Let* $0 < \alpha \leq 1$ *and* $r^{\alpha} < 0$ *for all* $r < 0$ *, then*

$$\left({}_{x_0}D_x^{\alpha}\right)[f(x)](x_0) = \Gamma(\alpha+1) \cdot \lim_{x \to x_0} \frac{f(x)-f(x_0)}{(x-x_0)^{\alpha}}. \tag{13}$$

Theorem 2.10 (fractional Fermat's theorem): *Suppose that* $0 < \alpha \leq 1$ *and* $r^{\alpha} < 0$ *for all* $r < 0$ *. If* x_0 *is an extreme point of* α*-fractional function* f *and* $\left({}_{x_0}D_x^{\alpha}\right)[f(x)](x_0)$ *exists, then* $\left({}_{x_0}D_x^{\alpha}\right)[f(x)](x_0) = 0$.

Proof Since $\left({}_{x_0}D_x^{\alpha}\right)[f(x)](x_0)$ exists, by Theorem 2.11, we know that $\lim_{x \to x_0} \frac{f(x)-f(x_0)}{(x-x_0)^{\alpha}}$ exists. If x_0 is a local maximum point of f, that is, $f(x_0) \geq f(x)$ on some neighbourhood of x_0. Then $\frac{f(x)-f(x_0)}{(x-x_0)^{\alpha}} \leq 0$ if $x \geq x_0$. It follows that $\lim_{x \to x_0^+} \frac{f(x)-f(x_0)}{(x-x_0)^{\alpha}} \leq 0$. On the other hand, if $x \leq x_0$, then $\frac{f(x)-f(x_0)}{(x-x_0)^{\alpha}} \geq 0$, and hence $\lim_{x \to x_0^-} \frac{f(x)-f(x_0)}{(x-x_0)^{\alpha}} \geq 0$. Therefore, $\lim_{x \to x_0} \frac{f(x)-f(x_0)}{(x-x_0)^{\alpha}} = 0$. Using Theorem 2.11 yields $\left({}_{x_0}D_x^{\alpha}\right)[f(x)](x_0) = 0$. The case that x_0 is a local minimum point of f can be proved in the same way. Q.e.d.

Theorem 2.11 (fractional Rolle's theorem): *Assume that* $0 < \alpha \leq 1$ *and* $r^{\alpha} < 0$ *for all* $r < 0$ *. If* f *is continuous on closed interval* $[a, b]$ *and* α*-fractional differentiable on open interval* (a, b) *with* $f(a) = f(b)$*, then there exists* $\xi \in (a, b)$ *such that* $\left({}_{a}D_x^{\alpha}\right)[f(x)](\xi) = 0$.

Proof Since f is continuous on closed interval $[a, b]$, f must have a maximum value M and a minimum value m on $[a, b]$. If $M = m$, then f is a constant function, and hence $\left({}_{a}D_x^{\alpha}\right)[f(x)](\xi) = 0$ for all $\xi \in (a, b)$. Without loss of generality, we may assume $M > m$. Since $f(a) = f(b)$, it follows that there is $\xi \in (a, b)$ such that $f(\xi) = M$. And hence ξ is an extreme point of f. By fractional Fermat's theorem, $\left({}_{a}D_x^{\alpha}\right)[f(x)](\xi) = 0$. Q.e.d.

Using fractional Rolle's theorem, we can obtain the following major result of this paper.

Theorem 2.12 (fractional mean value theorem): *Assume that* $0 < \alpha \leq 1$ *and* $r^{\alpha} < 0$ *for all* $r < 0$ *. If* f *is continuous on closed interval* $[a, b]$ *and is* α*-fractional differentiable on open interval* (a, b)*, then there exists* $\xi \in (a, b)$ *such that*

$$f(b) - f(a) = \frac{\left({}_{a}D_x^{\alpha}\right)[f(x)](\xi)}{\Gamma(\alpha+1)} \cdot (b-a)^{\alpha}. \tag{14}$$

Proof Let

$$g(x) = f(x) - \left[f(a) + \frac{f(b)-f(a)}{(b-a)^{\alpha}} (x-a)^{\alpha}\right]. \tag{15}$$

Since

$$\left({}_{a}D_x^{\alpha}\right)[g(x)] = \left({}_{a}D_x^{\alpha}\right)[f(x)] - \Gamma(\alpha+1) \cdot \frac{f(b)-f(a)}{(b-a)^{\alpha}}, \tag{16}$$

and

$$g(a) = f(a) - \left[f(a) + \frac{f(b)-f(a)}{(b-a)^{\alpha}} (a-a)^{\alpha}\right] = 0, \tag{17}$$

$$g(b) = f(b) - \left[f(a) + \frac{f(b)-f(a)}{(b-a)^{\alpha}} (b-a)^{\alpha}\right] = 0, \tag{18}$$

it follows from fractional Rolle's theorem that there is $\xi \in (a, b)$ such that $\left({}_{a}D_x^{\alpha}\right)[g(x)](\xi) = 0$. And hence,

$$\left({}_{a}D_x^{\alpha}\right)[f(x)](\xi) = \Gamma(\alpha+1) \cdot \frac{f(b)-f(a)}{(b-a)^{\alpha}}. \tag{19}$$

Therefore,

$$f(b) - f(a) = \frac{\left({}_{a}D_x^{\alpha}\right)[f(x)](\xi)}{\Gamma(\alpha+1)} \cdot (b-a)^{\alpha}.$$ Q.e.d.

Corollary 2.13: *Let* $0 < \alpha \leq 1$ *and* $r^{\alpha} < 0$ *for all* $r < 0$ *. If* f *is* α*-fractional differentiable on open interval* (a, b) *such that* $\left({}_{a}D_x^{\alpha}\right)[f(x)] = 0$ *for all* $x \in (a, b)$*. Then* f *is a constant function on* (a, b).

Proof If f is not a constant function on (a,b), then there exist x_1, x_2 such that

$$a < x_1 < x_2 < b \quad \text{and} \quad f(x_1) \neq f(x_2). \tag{20}$$

By fractional mean value theorem, we obtain

$$\left({}_{x_1}D_x^{\alpha}\right)[f(x)](\xi) = \Gamma(\alpha+1)\cdot\frac{f(x_2)-f(x_1)}{(x_2-x_1)^{\alpha}} \tag{21}$$

for some $\xi \in (x_1, x_2)$.

Therefore,

$$\left({}_{x_1}D_x^{\alpha}\right)[f(x)](\xi) \neq 0, \tag{22}$$

a contradiction. Q.e.d.

III. APPLICATIONS

Example 3.1: Suppose that $0 < \alpha \le 1$, a, b are real numbers, and $r^{\alpha} < 0$ for all $r < 0$. Let $f(x) = sin_{\alpha}(x^{\alpha})$. Using fractional mean value theorem yields

$$sin_{\alpha}(b^{\alpha}) - sin_{\alpha}(a^{\alpha}) = \frac{\left({}_{a}D_x^{\alpha}\right)[f(x)](\xi)}{\Gamma(\alpha+1)}(b-a)^{\alpha} \tag{23}$$

for some $\xi \in (a,b)$.

Since $\left|\frac{\left({}_{a}D_x^{\alpha}\right)[f(x)](\xi)}{\Gamma(\alpha+1)}\right| < K$ for some constant K. It follows that

$$|sin_{\alpha}(b^{\alpha}) - sin_{\alpha}(a^{\alpha})| \le K|b-a|^{\alpha}. \tag{24}$$

That is, $sin_{\alpha}(x^{\alpha})$ is a Hölder continuous function with exponent α.

Proposition 3.2: Let $0 < \alpha \le 1$, $x > 0$, and $r^{\alpha} < 0$ for all $r < 0$. Then

$$\frac{1}{\Gamma(\alpha+1)}x^{\alpha}\otimes\left(1+\frac{1}{\Gamma(\alpha+1)}x^{\alpha}\right)^{\otimes -1} < Ln_{\alpha}(1+x^{\alpha}) < \frac{1}{\Gamma(\alpha+1)}x^{\alpha}. \tag{25}$$

Proof Let $f(t) = Ln_{\alpha}(1+t^{\alpha})$. Since

$$\left({}_{0}D_t^{\alpha}\right)[f(t)] = \left(1+\frac{1}{\Gamma(\alpha+1)}t^{\alpha}\right)^{\otimes -1} \tag{26}$$

and $f(t)$ satisfies the conditions of fractional mean value theorem on closed interval $[0,x]$, it follows that

$$f(x) = f(x) - f(0) = \frac{\left({}_{0}D_t^{\alpha}\right)[f(t)](\xi)}{\Gamma(\alpha+1)}\cdot x^{\alpha} = \left(1+\frac{1}{\Gamma(\alpha+1)}\xi^{\alpha}\right)^{\otimes -1}\cdot\frac{1}{\Gamma(\alpha+1)}x^{\alpha} \tag{27}$$

for some $\xi \in (0,x)$.

Therefore,

$$Ln_{\alpha}(1+x^{\alpha}) = \left(1+\frac{1}{\Gamma(\alpha+1)}\xi^{\alpha}\right)^{\otimes -1}\cdot\frac{1}{\Gamma(\alpha+1)}x^{\alpha}. \tag{28}$$

Since $0 < \xi < x$, it follows that

$$\frac{1}{\Gamma(\alpha+1)}x^{\alpha}\otimes\left(1+\frac{1}{\Gamma(\alpha+1)}x^{\alpha}\right)^{\otimes -1} < \frac{1}{\Gamma(\alpha+1)}x^{\alpha}\cdot\left(1+\frac{1}{\Gamma(\alpha+1)}\xi^{\alpha}\right)^{\otimes -1} < \frac{1}{\Gamma(\alpha+1)}x^{\alpha}. \tag{29}$$

That is,

$$\frac{1}{\Gamma(\alpha+1)}x^{\alpha}\otimes\left(1+\frac{1}{\Gamma(\alpha+1)}x^{\alpha}\right)^{\otimes -1} < Ln_{\alpha}(1+x^{\alpha}) < \frac{1}{\Gamma(\alpha+1)}x^{\alpha}.$$ Q.e.d.

IV. CONCLUSION

As mentioned above, we use fractional Fermat's theorem and fractional Rolle's theorem to prove fractional mean value theorem based on Jumarie's modified R-L fractional derivatives. The fractional mean value theorem is the theoretical basis of fractional differential calculus, and we hope that it can be widely used to solve many problems in fractional calculus. In the future, we will make use of the methods provided in this paper to extend our research fields to applied mathematics and fractional differential equations.

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