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A New Insight into Fractional Logistic Equation

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Abstract: **In this article, based on Jumarie type of modified Riemann-Liouville (R-L) fractional derivatives, we make use of a new multiplication and some techniques include separation of variables, partial fraction integration and chain rule for fractional derivatives to obtain the closed form solution of the fractional logistic equation we defined. In fact, the fractional logistic equation is a generalization of the classical logistic equation. Moreover, the Mittag-Leffler function plays an important role in this paper.**

Keywords: **Jumarie type of modified R-L fractional derivatives, new multiplication, closed form solution, fractional logistic equation, Mittag-Leffler function.**

I. INTRODUCTION

For centuries, mathematics has played a vital role in the development of human civilization, because in other fields, it allows the description and prediction of events in the real world, through mathematical representations. In this regard, it is reasonable to emphasize the importance of differential calculus and integral calculus for the study of many of the laws of nature. Fractional calculus is the study of derivatives and integrals of arbitrary orders. For a long time, the theory of fractional calculus developed only as a theoretical field of mathematics. However, in the last decades, it was shown that some fractional operators can better describe some complex physical phenomena, so fractional calculus has been paid more and more attention by mathematicians. On the other hand, physicists and engineers are also very interested in the applications of this nice theory. Many real life phenomena have been described using fractional differential equations, such as viscoelasticy, continuum mechanics, optimal control, hydrologic modelling, variational problems, fluid mechanics, finance, and others [1-17]. Furthermore, the applications of fractional calculus to fractional differential equations can refer to [18-25].

Logistic equation is a famous population growth model introduced by mathematical biologist Pierre Francois Verhulst [26]. It is the extension of Malthus population model. The logistic population model is considered as an important type of nonlinear differential equations because it can be widely used in biology, medicine, economics and management. The classical logistic (or Verhulst's) equation is the nonlinear initial value problem:

$$\begin{cases} \frac{d}{dt}N(t) = kN(t)\left(1 - \frac{1}{N_{max}}N(t)\right), \quad t \geq 0 \\ N(0) = N_0 \end{cases}, \tag{1}$$

where $N(t)$ denotes the population at time t, $k > 0$ is the rate of maximum population growth, $N_0 > 0$ is the population at time $t = 0$, and N_{max} is the carrying capacity, i.e., the maximum attainable value of population. By dividing both sides of Eq. (1) by N_{max} and defining $u(t) = \frac{1}{N_{max}}N(t)$ as the normalization of population to its maximum attainable value, we obtain the differential equation with initial value:

$$\begin{cases} \frac{d}{dt}u(t) = ku(t)\big(1 - u(t)\big), \quad t \geq 0 \\ u(0) = u_0 \end{cases}. \tag{2}$$

For each realization of $k > 0$, Eq. (2) has an exact closed form solution:

$$u(t) = \frac{u_0}{u_0+(1-u_0)e^{-kt}}, \qquad t \geq 0. \tag{3}$$

The fractional logistic equation is a generalization of the classical model that is obtained when replacing the first order derivative by a fractional derivative of order α, $0 < \alpha \leq 1$. The solution to the fractional logistic equation has received much attention from many researchers in the field of fractional calculus, and some attempts were made to determine the exact, analytical or approximation solution [27-35]. In this paper, based on Jumarie's modified Riemann- Liouville fractional derivatives, a new multiplication is proposed, and the closed form solution of the fractional logistic equation is obtained by using the methods of separation of variables, partial fractional integration and chain rule for fractional derivatives. The exponential function plays a fundamental role in mathematical analysis and it is really useful in the theory of integer order differential equations. In the case of fractional order, it loses some beautiful properties and Mittag-Leffler function appears as its natural substitution.

II. PRELIMINARIES AND METHODS

At first, we introduce the definition of fractional derivative used in this paper and provide some basic properties as follows:

Definition 2.1: Let α be a real number and p be a positive integer. The Jumarie type modified R-L fractional derivatives ([36]) is defined as

$$ {}_aD_t^{\alpha}[f(t)] = \begin{cases} \frac{1}{\Gamma(-\alpha)}\int_a^t (t-\tau)^{-\alpha-1}f(\tau)d\tau, & \text{if } \alpha < 0 \\ \frac{1}{\Gamma(1-\alpha)}\frac{d}{dt}\int_a^t (t-\tau)^{-\alpha}[f(\tau)-f(a)]\,d\tau & \text{if } 0 \leq \alpha < 1 \\ \frac{d^p}{dt^p}\left({}_aD_t^{\alpha-p}\right)[f(t)], & \text{if } p \leq \alpha < p+1 \end{cases}, \tag{4}$$

where $\Gamma(y) = \int_0^{\infty} s^{y-1}e^{-s}ds$ is the gamma function defined on $y > 0$. On the other hand, we define the α-fractional integral of $f(t)$ as $\left({}_aI_t^{\alpha}\right)[f(t)] = \left({}_aD_t^{-\alpha}\right)[f(t)]$, where $\alpha > 0$ and $f(t)$ is called α-fractional integrable function.

Proposition 2.2 ([37])**:** *If α, β, c are real numbers and $\beta \geq \alpha > 0$, then*

$$ {}_0D_t^{\alpha}\left[t^{\beta}\right] = \frac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha+1)}t^{\beta-\alpha}, \tag{5}$$

and

$$ {}_0D_t^{\alpha}[c] = 0. \tag{6}$$

Definition 2.3 ([38])**:** The function $E_{\alpha}(z)$ is named after the great Swedish mathematician Gösta Mittag-Leffler (1846-1927) who defined it as a power series given by

$$E_{\alpha}(z) = \sum_{n=0}^{\infty}\frac{z^n}{\Gamma(n\alpha+1)}, \tag{7}$$

where α is a real number, $\alpha > 0$, and z is a complex variable. $E_{\alpha}(t) = \sum_{n=0}^{\infty}\frac{t^{n\alpha}}{\Gamma(n\alpha+1)}$ is called the α-fractional exponential function.

Next, we introduce a new multiplication of fractional functions. We continue with this definition mentioned in [39].

Definition 2.4 ([39])**:** Suppose that λ, μ, z are complex numbers, $0 < \alpha \leq 1$, j, l, n are non-negative integers, and a_n, b_n are real numbers, $p_n(z) = \frac{1}{\Gamma(n\alpha+1)}z^n$ for all n. The $\otimes$ multiplication is defined as

$$p_j(\lambda t^{\alpha}) \otimes p_l(\mu s^{\alpha}) = \frac{1}{\Gamma(j\alpha+1)}(\lambda t^{\alpha})^j \otimes \frac{1}{\Gamma(l\alpha+1)}(\mu s^{\alpha})^l = \frac{1}{\Gamma((j+l)\alpha+1)}\binom{j+l}{j}(\lambda t^{\alpha})^j(\mu s^{\alpha})^l, \tag{8}$$

where $\binom{j+l}{j} = \frac{(j+l)!}{j!\,l!}$.

Let $f(\lambda t^{\alpha})$ and $g(\mu s^{\alpha})$ be two fractional functions,

$$f(\lambda t^{\alpha}) = \sum_{n=0}^{\infty} a_n\, p_n(\lambda t^{\alpha}) = \sum_{n=0}^{\infty}\frac{a_n}{\Gamma(n\alpha+1)}(\lambda t^{\alpha})^n, \tag{9}$$

$$g(\mu s^{\alpha}) = \sum_{n=0}^{\infty} b_n\, p_n(\mu s^{\alpha}) = \sum_{n=0}^{\infty} \frac{b_n}{\Gamma(n\alpha+1)} (\mu s^{\alpha})^n. \tag{10}$$

Then we define

$$f(\lambda t^{\alpha}) \otimes g(\mu s^{\alpha}) = \sum_{n=0}^{\infty} a_n\, p_n(\lambda t^{\alpha}) \otimes \sum_{n=0}^{\infty} b_n\, p_n(\mu s^{\alpha})$$
$$= \sum_{n=0}^{\infty}\left(\sum_{m=0}^{n} a_{n-m} b_m p_{n-m}(\lambda t^{\alpha}) \otimes p_m(\mu s^{\alpha})\right). \tag{11}$$

Proposition 2.5: $f(\lambda t^{\alpha}) \otimes g(\mu s^{\alpha}) = \sum_{n=0}^{\infty} \frac{1}{\Gamma(n\alpha+1)} \sum_{m=0}^{n} \binom{n}{m} a_{n-m} b_m\, (\lambda t^{\alpha})^{n-m} (\mu s^{\alpha})^m.$ (12)

Definition 2.6: Suppose that $\left(f(\lambda t^{\alpha})\right)^{\otimes n} = f(\lambda t^{\alpha}) \otimes \cdots \otimes f(\lambda t^{\alpha})$ is the n times product of the fractional function $f(\lambda t^{\alpha})$. If $f(\lambda t^{\alpha}) \otimes g(\lambda t^{\alpha}) = 1$, then $g(\lambda t^{\alpha})$ is called the $\otimes$ reciprocal of $f(\lambda t^{\alpha})$, and denoted as $\left(f(\lambda t^{\alpha})\right)^{\otimes -1}$.

Theorem 2.7 (chain rule for fractional derivatives) ([39])**:** *If* $f(z) = \sum_{n=0}^{\infty} a_n z^n$, $g_{\alpha}(\mu t^{\alpha}) = \sum_{n=0}^{\infty} b_n\, p_n(\mu t^{\alpha})$. *Let* $f_{\otimes\alpha}\left(g_{\alpha}(\mu t^{\alpha})\right) = \sum_{n=0}^{\infty} a_n \left(g_{\alpha}(\mu t^{\alpha})\right)^{\otimes n}$ *and* $f'_{\otimes\alpha}\left(g_{\alpha}(\mu t^{\alpha})\right) = \sum_{n=1}^{\infty} n a_n \left(g_{\alpha}(\mu t^{\alpha})\right)^{\otimes(n-1)}$, *then*

$$\left({}_0D_t^{\alpha}\right)\left[f_{\otimes\alpha}\left(g_{\alpha}(\mu t^{\alpha})\right)\right] = f'_{\otimes\alpha}\left(g_{\alpha}(\mu t^{\alpha})\right) \otimes \left({}_0D_t^{\alpha}\right)\left[g_{\alpha}(\mu t^{\alpha})\right]. \tag{13}$$

III. MAJOR RESULT

The main purpose of this section is to introduce the fractional logistic equation we defined, and derive the closed form solution of this equation.

Theorem 3.1 *Let* $0 < \alpha \le 1$, *and* $k > 0$. *The fractional logistic equation with initial value:*

$$\begin{cases} \left({}_0D_t^{\alpha}\right)[u_{\alpha}(t)] = k u_{\alpha}(t) \otimes \left(1 - u_{\alpha}(t)\right), \quad t \ge 0 \\ u_{\alpha}(0) = u_0 \end{cases} \tag{14}$$

has an exact solution

$$u_{\alpha}(t) = u_0\left(u_0 + (1-u_0)E_{\alpha}(-kt)\right)^{\otimes -1}, \qquad t \ge 0. \tag{15}$$

Proof By separation of variables, we have

$$\left({}_{u_0}I_u^{\alpha}\right)\left[\left(u_{\alpha} \otimes (1-u_{\alpha})\right)^{\otimes -1}\right] = \left({}_0I_t^{\alpha}\right)[k]\,. \tag{16}$$

Using partial fraction integral method yields

$$\left(u_{\alpha}(t) - 1\right) \otimes u_{\alpha}(t)^{\otimes -1} = C E_{\alpha}(-kt)\,, \tag{17}$$

where C is a constant.

Since $u_{\alpha}(0) = u_0$, it follows that $C = \frac{u_0 - 1}{u_0}$. And hence

$$u_{\alpha}(t)^{\otimes -1} = 1 - \frac{u_0 - 1}{u_0} E_{\alpha}(-kt) = \frac{u_0 + (1-u_0)E_{\alpha}(-kt)}{u_0}. \tag{18}$$

Therefore,

$$u_{\alpha}(t) = u_0\left(u_0 + (1-u_0)E_{\alpha}(-kt)\right)^{\otimes -1}$$ Q.e.d.

Remark 3.2: We can verify the correctness of Theorem 3.1 directly. By chain rule for fractional derivatives, the fractional derivative of Eq. (15)

$$\left({}_0D_t^{\alpha}\right)[u_{\alpha}(t)] = \left({}_0D_t^{\alpha}\right)\left[u_0\left(u_0 + (1-u_0)E_{\alpha}(-kt)\right)^{\otimes -1}\right]$$
$$= k u_0\left(u_0 + (1-u_0)E_{\alpha}(-kt)\right)^{\otimes -2} \otimes \left((1-u_0)E_{\alpha}(-kt)\right)$$
$$= k u_{\alpha}(t) \otimes \left(1 - u_{\alpha}(t)\right).$$

IV. CONCLUSION

From the above discussion, we know that the new multiplication, separation of variables, and chain rule for fractional derivatives play important roles in this article. In fact, these methods are widely used and can easily solve many problems of fractional calculus and fractional differential equations. In addition, the fractional logistic equation is a generalization of the classical logistic equation. In the future, we will use the Jumarie type of modified R-L fractional derivatives, the new multiplication, and Mittag-Leffler function to expand our research area to applied science and engineering mathematics.

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