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Application of Fractional Bernstein Polynomial

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Abstract: **This paper uses the fractional Bernstein theorem to prove the fractional Weierstrass's approximation theorem. And hence, we obtain that the set of fractional functions is dense in the set of continuous functions on a closed interval. The fractional Bernstein polynomial plays an important role in this article, which is the generalization of classical Bernstein polynomial.**

Keywords: **fractional Bernstein theorem, fractional Weierstrass's approximation theorem, fractional Bernstein polynomial.**

I. INTRODUCTION

Bernstein polynomial is the polynomial named after Russian mathematician Bernstein. It is a remarkable family of polynomials associated to any given function on the unit interval. If f is continuous on $[0,1]$, its n-th Bernstein polynomial is defined by

$$B_n f(x) = \sum_{k=0}^{n} f\left(\frac{k}{n}\right)\binom{n}{k} x^k (1-x)^{n-k}. \tag{1}$$

Bernstein showed in [1] that $B_n f(x)$ converges uniformly to $f(x)$ on $[0,1]$, thus giving a constructive proof of the Weierstrass's approximation theorem, which stated that any continuous function defined on closed interval can be approximated by polynomials. One might wonder why Bernstein created new polynomials for this purpose, instead of using polynomials that were already known to mathematics. Taylor polynomials are not appropriate; for even setting aside questions of convergence, they are applicable only to functions that are infinitely differentiable, and not to all continuous functions.

On the other hand, the concept of fractional calculus originated from L'Hospital to Leibniz in 1695. One of the questions he asked was, "what will be the result about $\frac{d^{1/2}x}{dx^{1/2}}$? ".But until 1819, Lacroix gave the simplest result of fractional calculus for the first time that $\frac{d^{1/2}x}{dx^{1/2}} = \frac{2}{\sqrt{\pi}} x^{1/2}$. However, unlike the derivative of integer order, the fractional derivative is not unique. There are some different definitions of fractional derivative: Riemann-Liouville (R-L), Grunwald-Letnikov, Caputo, Miller Ross sequence, and Jumarie type of modified R-L fractional derivatives. In recent decades, researchers have found that the fractional calculus operator has nonlocality property, that is, the next state of a system depends not only on its current state, but also on its historical state starting from the initial time. Therefore, it is very suitable to describe the material with memory and genetic properties in the real world. Compared with the integer order model, the fractional order model is perfect. Moreover, the analysis shows that the fractional order model is more practical than the integer order model. In addition, fractional calculus has wide applications in viscoelasticity, quantum mechanics, electromagnetism, electrochemistry, signal and image processing, vibration and oscillation, biology continuum mechanics, electrical engineering, fluid science, control theory, dynamics, finance, and so on [2-23]. For more details and applications of Jumarie's modified R-L fractional derivative, we can see [24-39].

In this paper, we prove the fractional Bernstein theorem, and hence the fractional Weierstrass's approximation theorem can be obtained. In other words, we show that the set of α-fractional polynomials is dense in $C[a,b]$ (the set of all continuous functions defined on closed interval $[a,b]$) for any $0<\alpha\le 1$. In fact, the fractional Bernstein polynomial plays an important role in this study, which is the generalization of classical Bernstein polynomial.

II. PRELIMINARIES

To prove the main results in this paper, we need the following lemmas.

Lemma 2.1: *Let* $0<\alpha\le 1$. *For any real number* x, *and integer* $n\ge 0$, *we have*

$$\sum_{k=0}^{n}\binom{n}{k}(x^\alpha)^k(1-x^\alpha)^{n-k}=1. \tag{2}$$

Proof $\sum_{k=0}^{n}\binom{n}{k}(x^\alpha)^k(1-x^\alpha)^{n-k} = (x^\alpha+1-x^\alpha)^n=(1)^n=1.$ Q.e.d.

Lemma 2.2: $\sum_{k=0}^{n}k\binom{n}{k}(x^\alpha)^k(1-x^\alpha)^{n-k}=nx^\alpha.$ (3)

Proof Since $k\binom{n}{k}=n\binom{n-1}{k-1}$ for $k\ge 1$. We have

$$\sum_{k=0}^{n}k\binom{n}{k}(x^\alpha)^k(1-x^\alpha)^{n-k}$$
$$=n\sum_{k=1}^{n}\binom{n-1}{k-1}(x^\alpha)^k(1-x^\alpha)^{n-k}$$
$$=nx^\alpha\cdot\sum_{m=0}^{n-1}\binom{n-1}{m}(x^\alpha)^m(1-x^\alpha)^{n-1-m}$$
$$=nx^\alpha\ .$$

Q.e.d.

Lemma 2.3: $\sum_{k=0}^{n}k(k-1)\binom{n}{k}(x^\alpha)^k(1-x^\alpha)^{n-k}=n\,(n-1)x^{2\alpha}.$ (4)

Proof Since for $k\ge 2$, $k(k-1)\binom{n}{k}=n(n-1)\binom{n-2}{k-2}$. It follows that

$$\sum_{k=0}^{n}k(k-1)\binom{n}{k}(x^\alpha)^k(1-x^\alpha)^{n-k}$$
$$=n(n-1)(x^\alpha)^2\cdot\sum_{l=0}^{n-2}\binom{n-2}{l}(x^\alpha)^l(1-x^\alpha)^{n-2-l}$$
$$=n(n-1)x^{2\alpha}.$$

Q.e.d.

Lemma 2.4: $\sum_{k=0}^{n}k^2\binom{n}{k}(x^\alpha)^k(1-x^\alpha)^{n-k}=n^2x^{2\alpha}+nx^\alpha(1-x^\alpha).$ (5)

Proof Adding the results in Eqs. (3) and (4), we obtain

$$\sum_{k=0}^{n}k^2\binom{n}{k}(x^\alpha)^k(1-x^\alpha)^{n-k}$$
$$=n(n-1)x^{2\alpha}+nx^\alpha$$
$$=n^2x^{2\alpha}+nx^\alpha(1-x^\alpha).$$

Q.e.d.

Lemma 2.5: $\sum_{k=0}^{n}(nx^\alpha-k)^2\binom{n}{k}(x^\alpha)^k(1-x^\alpha)^{n-k}=n\,x^\alpha(1-x^\alpha)\le\frac{n}{4}.$ (6)

Proof Since $(nx^\alpha-k)^2=n^2x^{2\alpha}-2nkx^\alpha+k^2$. Using Lemmas 2.2 and 2.4 yields

$$\sum_{k=0}^{n}(nx^\alpha-k)^2\binom{n}{k}(x^\alpha)^k(1-x^\alpha)^{n-k}$$
$$=n^2x^{2\alpha}-2n^2x^{2\alpha}+n^2x^{2\alpha}+nx^\alpha(1-x^\alpha)$$
$$=nx^\alpha(1-x^\alpha).$$

And

$$nx^\alpha(1-x^\alpha)=n\left[-\left(x^\alpha-\frac{1}{2}\right)^2+\frac{1}{4}\right]\le\frac{n}{4}.$$

Q.e.d.

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III. MAJOR RESULTS

The followings are the major theorems of this article.

Theorem 3.1 (fractional Bernstein theorem): *Let* $0 < \alpha \leq 1$. *If* f *is continuous on closed interval* $[0,1]$, *then the* α-*fractional Bernstein polynomial of* f ,

$$B_{\alpha,n}f(x) = \sum_{k=0}^{n} f\left(\left(\frac{k}{n}\right)^{\frac{1}{\alpha}}\right)\binom{n}{k}(x^{\alpha})^{k}(1-x^{\alpha})^{n-k} \tag{7}$$

is uniformly convergent to $f(x)$ *on* $[0,1]$.

Proof We may assume that f is not identically zero and let $K = \max_{x\in[0,1]}|f(x)|$. Since f is uniformly continuous on $[0,1]$, it follows that for any $\varepsilon > 0$, there exists $\delta_1 > 0$ such that

$$|u - w| < \delta_1 \text{ implies that } |f(u) - f(w)| < \frac{\varepsilon}{2} \quad \text{for all } u, w \in [0,1]. \tag{8}$$

For a fixed $x \in [0,1]$, since $z^{\frac{1}{\alpha}}$ is a continuous function on $[0,1]$, it follows that there exists $\delta > 0$ such that

$$|y - x^{\alpha}| < \delta \text{ implies that } \left|y^{\frac{1}{\alpha}} - x\right| < \delta_1. \tag{9}$$

And hence,

$$\left|f\left(y^{\frac{1}{\alpha}}\right) - f(x)\right| < \frac{\varepsilon}{2}. \tag{10}$$

On the other hand, using Lemma 3.1 we have

$$f(x) = \sum_{k=0}^{n} f(x)\binom{n}{k}(x^{\alpha})^{k}(1-x^{\alpha})^{n-k}. \tag{11}$$

Thus,

$$\left|B_{\alpha,n}f(x) - f(x)\right| \leq \sum_{k=0}^{n}\left|f\left(\left(\frac{k}{n}\right)^{\frac{1}{\alpha}}\right) - f(x)\right| \cdot \binom{n}{k}(x^{\alpha})^{k}(1-x^{\alpha})^{n-k}. \tag{12}$$

To estimate this sum, we divide the set $\{0,1,2,\cdots,n\}$ into two sets C and D:

$$k \in C \text{ if } \left|\frac{k}{n} - x^{\alpha}\right| < \delta \text{ and } k \in D \text{ if } \left|\frac{k}{n} - x^{\alpha}\right| \geq \delta.$$

Case 1: If $k \in C$, then $\left|\frac{k}{n} - x^{\alpha}\right| < \delta$, it follows that $\left|\left(\frac{k}{n}\right)^{\frac{1}{\alpha}} - x\right| < \delta_1$, and hence $\left|f\left(\left(\frac{k}{n}\right)^{\frac{1}{\alpha}}\right) - f(x)\right| < \frac{\varepsilon}{2}$. Therefore,

$$\sum_{k\in C}\left|f\left(\left(\frac{k}{n}\right)^{\frac{1}{\alpha}}\right) - f(x)\right| \cdot \binom{n}{k}(x^{\alpha})^{k}(1-x^{\alpha})^{n-k} < \sum_{k\in C}\frac{\varepsilon}{2}\cdot\binom{n}{k}(x^{\alpha})^{k}(1-x^{\alpha})^{n-k} < \frac{\varepsilon}{2}. \tag{13}$$

Case 2: If $k \in D$, that is, $\left|\frac{k}{n} - x^{\alpha}\right| \geq \delta$, then $(nx^{\alpha} - k)^2 \geq n^2\delta^2$. Let $M = \frac{K}{\varepsilon\delta^2}$. Thus, if $n > M$, by Lemma 3.5, we have

$$\begin{aligned}
&\sum_{k\in D}\left|f\left(\left(\frac{k}{n}\right)^{\frac{1}{\alpha}}\right) - f(x)\right| \cdot \binom{n}{k}(x^{\alpha})^{k}(1-x^{\alpha})^{n-k} \\
&\leq \frac{2K}{n^2\delta^2}\sum_{k\in D}(nx^{\alpha} - k)^2\binom{n}{k}(x^{\alpha})^{k}(1-x^{\alpha})^{n-k} \\
&\leq \frac{2K}{n^2\delta^2}\left(\frac{n}{4}\right) \\
&= \frac{K}{2n\delta^2} \\
&< \frac{K}{2M\delta^2} \\
&= \frac{\varepsilon}{2}.
\end{aligned} \tag{14}$$

Therefore, by Eqs. (13), (14), we have

$$\left|B_{\alpha,n}f(x) - f(x)\right| < \varepsilon \text{ for all } x \in [0,1] \text{ and all } n > M. \quad (15)$$

Thus, $B_{\alpha,n}f(x)$ is uniformly convergent to f(x) on $[0,1]$. Q.e.d.

Theorem 3.2 (fractional Weierstrass's approximation theorem): *Suppose that* $0 < \alpha \leq 1$. *Every continuous function defined on a closed interval* $[a, b]$ *can be uniformly approximated by* α*- fractional polynomials on* $[a, b]$.

Proof Let $f(x)$ be a continuous function on $[a, b]$ and let $\rho(x) = a + (b - a)x$. Then $F(x) = f\big(\rho(x)\big)$ is a continuous function on $[0,1]$. By fractional Bernstein theorem, the α-fractional Bernstein polynomial of $F(x)$,

$$B_{\alpha,n}F(x) = \sum_{k=0}^{n} F\left(\left(\frac{k}{n}\right)^{\frac{1}{\alpha}}\right)\binom{n}{k}(x^{\alpha})^{k}(1 - x^{\alpha})^{n-k} \quad (16)$$

is uniformly convergent to $F(x)$ on $[0,1]$.

Since $f(x) = F\left(\frac{x-a}{b-a}\right)$, it follows that the α-fractional polynomial

$$\sum_{k=0}^{n} F\left(\left(\frac{k}{n}\right)^{\frac{1}{\alpha}}\right)\binom{n}{k}\left(\left(\frac{x-a}{b-a}\right)^{\alpha}\right)^{k}\left(1 - \left(\frac{x-a}{b-a}\right)^{\alpha}\right)^{n-k}$$

$$= \frac{1}{(b-a)^{n\alpha}} \cdot \sum_{k=0}^{n} f\left(a + \left(\frac{k}{n}\right)^{\frac{1}{\alpha}}(b - a)\right)\binom{n}{k}(x - a)^{k\alpha}[(b - a)^{\alpha} - (x - a)^{\alpha}]^{n-k}. \quad (17)$$

And it is uniformly convergent to $f(x)$ on $[a, b]$. Q.e.d.

IV. CONCLUSION

From the above discussion, we know that the fractional Weierstrass's approximation theorem is the generalization of the classical one in mathematical analysis. In addition, the fractional Bernstein polynomial plays a vital role in the proof of this theorem. In the future, we will take advantage of fractional Bernstein polynomial to study the problems in applied mathematics and fractional calculus.

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